ON LARGE DEVIATIONS OF ADDITIVE FUNCTIONS

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ABSTRACT. We prove that if two additive functions (from a certain class) take large values with roughly the same probability then they must be identical. The Kac-Kubilius model suggests that the distribution of values of a given additive function can be modeled by a sum of random variables. We show that the model is accurate (in a large deviation sense) when one is looking at values of the additive function around its mean, but fails, by "a constant multiple", for large values of the additive function. We believe that this phenomenon arises, because the model breaks down for the values of the additive function on the "large" primes.

In the second part of the paper, we are motivated by a question of Elliott, to understand how much the distribution of values of the additive function on primes determines, and is determined by, the distribution of values of the additive function on all of the integers. For example, our main theorem implies that a positive, strongly additive function is roughly Poisson distributed on the integers if and only if it is $1 + o(1)$ or $o(1)$ on almost all primes.

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1. Introduction

Let g be a strongly additive function. According to Mark Kac, the distribution of the $g(n)$'s (with $n \leqslant x$ and x large) can be predicted by studying the random variable

$$\sum_{p \leqslant x} g(p) X_p \tag{1.1}$$

In (1.1) the X_p's are independent random variables with $\mathbb{P}(X_p = 1) = 1/p$ and $\mathbb{P}(X_p = 0) = 1 - 1/p$. Thus, we expect the $g(n)$'s to cluster around the mean $\mu(g;x)$ of (1.1) and within $O(\sigma(g;x))$. Here $\mu(g;x)$ and $\sigma^2(g;x)$ are respectively the mean and the variance of (1.1), so that

$$\mu(g;x) = \sum_{p \leqslant x} \frac{g(p)}{p} \quad \text{and} \quad \sigma^2(g;x) = \sum_{p \leqslant x} \frac{g(p)^2}{p} \cdot \left(1 - \frac{1}{p}\right)$$

The Erdös-Kac theorem (see [5], theorem 12.2), states that

$$\mathcal{D}_g(x;\Delta) := \frac{1}{x} \cdot \#\left\{ n \leqslant x : \frac{g(n) - \mu(g;x)}{\sigma(g;x)} \geqslant \Delta \right\} \sim \int_{\Delta}^{\infty} e^{-u^2/2} \cdot \frac{du}{\sqrt{2\pi}}$$

for $\Delta = O(1)$ and any reasonable additive function g. However it is known (see [14]) that this "normal approximation" fails, already when $\Delta = \sigma(g;x)^{1/3}$. In this range $\mathcal{D}_g(x;\Delta)$ is asymptotic to a constant $c < 1$ times the normal distribution. For larger values of Δ there is an ugly asymptotic (see [14] or [11]) comparing $\mathcal{D}_g(x;\Delta)$ to the normal law. Our first contribution is the observation that the ugly asymptotic can be recast in a more natural form. Namely, we have

$$\frac{1}{x}\cdot\#\left\{n\leqslant x:\frac{g(n)-\mu(g;x)}{\sigma(g;x)}\geqslant\Delta\right\}\sim\mathbb{P}\left(\sum_{p\leqslant x}\left[g(p)-\frac{1}{p}\right]X_p\geqslant\Delta\sigma(g;x)\right)\tag{1.2}$$

uniformly in $1\leqslant\Delta\leqslant o(\sigma(g;x))$, for instance for strongly additive functions g such that $0\leqslant g(p)\leqslant O(1)$ and $\sigma(g;x)\to\infty$. Since (1.2) is a natural extension of the Erdös-Kac, it is desirable to know if (1.2) holds for all additive functions for which the Erdös-Kac does (in the version of [5], theorem 12.2). We leave this open. Instead, we ask in which range (1.2) is no longer true, and by how much does (1.2) fail in that range? To answer this question we confine our attention to the class $\mathcal{C}$, defined below.

Definition. *An additive function g belongs to $\mathcal{C}$ if and only if*

- *The function g is strongly additive*[1] *and strictly positive.*
- *Given any $A>0$, we have for all $t\geqslant 0$ and $x\geqslant 2$,*

$$\frac{1}{\pi(x)}\sum_{\substack{p\leqslant x\\ g(p)\geqslant t}}1=O_A\left(e^{-At}\right)\tag{1.3}$$

- *There is a distribution function $\Psi(g;t)$ with non-zero second moment, such that for all $k>0$*

$$\frac{1}{\pi(x)}\sum_{\substack{p\leqslant x\\ g(p)\leqslant t}}1-\Psi(g;t)=O_k\left(\frac{1}{\log^k x}\right)\ \textit{uniformly in } t\in\mathbb{R}\tag{1.4}$$

Additive functions belonging to $\mathcal{C}$ are particularly well-behaved. Nonetheless, even for a $g\in\mathcal{C}$, the asymptotic (1.2) doesn't hold in the wider range $\Delta\asymp\sigma(g;x)$. Indeed, we prove that for any fixed $\delta>0$, uniformly in $1\leqslant\Delta\leqslant\delta\sigma(g;x)$,

$$\frac{1}{x}\#\left\{n\leqslant x:\frac{g(n)-\mu(g;x)}{\sigma(g;x)}\geqslant\Delta\right\}\sim\mathcal{A}\left(g;\frac{\Delta}{\sigma}\right)\mathbb{P}\left(\sum_{p\leqslant x}g(p)\left[X_p-\frac{1}{p}\right]\geqslant\Delta\sigma\right)\tag{1.5}$$

where $\mathcal{A}(g;z)$ is an analytic function depending only on $\Psi(g;\cdot)$ and σ stands for $\sigma(g;x)$. Most importantly $0<\mathcal{A}(g;x)\leqslant\mathcal{A}(g;0)=1$ for positive x, and $\mathcal{A}(g;x)$ is a strictly decreasing function, decaying to 0 as $x\to\infty$. For example when $g(n)=\omega(n)$, where $\omega(n)$ is the number of prime factors of n, we have $\mathcal{A}(g;z)=e^{-\gamma z}/\Gamma(1+z)$ (where γ is the Euler-Mascheroni constant). The appearance of the factor $\mathcal{A}(g;\cdot)$, is largely due to the large prime factors, and we state a precise conjecture explaining the phenomena, in the next section. In order to prove (1.5) we simply establish asymptotics for the left and right hand side of (1.5) and then compare them.

[1]That is $g(p^k)=g(p)$ for all primes p and integers $k\geq 1$, and $g(mn)=g(m)+g(n)$ whenever $(m,n)=1$.

Our main result is a "structure theorem", classifying additive functions in $\mathcal{C}$ in terms of the distribution of their large values.

Theorem 1.1. *(The "structure theorem") Let* $f, g \in \mathcal{C}$. *Suppose that* $\sigma(f;x) \sim \sigma(g;x)$ *and let* $\sigma := \sigma(x)$ *denote a function such that* $\sigma(f;x) \sim \sigma(x) \sim \sigma(g;x)$. *The asymptotic*

$$\frac{1}{x} \cdot \# \left\{ n \leqslant x : \frac{f(n) - \mu(f;x)}{\sigma(f;x)} \geqslant \Delta \right\} \sim \frac{1}{x} \cdot \# \left\{ n \leqslant x : \frac{g(n) - \mu(g;x)}{\sigma(g;x)} \geqslant \Delta \right\} \tag{1.6}$$

holds uniformly, in the range,

(1) $1 \leqslant \Delta \leqslant o(\sigma^{1/3})$ *– always (the distribution is normal)*
(2) $1 \leqslant \Delta \leqslant o(\sigma^{\alpha})$ *with an* $\alpha \in (1/3;1)$ *if and only if*

$$\int_{\mathbb{R}} t^k d\Psi(f;t) \quad = \quad \int_{\mathbb{R}} t^k d\Psi(g;t)$$

for all $k = 3, 4, \ldots, \rho(\alpha)$ *where* $\rho(\alpha) := \lceil (1+\alpha)/(1-\alpha) \rceil$
(3) $1 \leqslant \Delta \leqslant o(\sigma)$ *if and only if* $\Psi(f;t) = \Psi(g;t)$
(4) $1 \leqslant \Delta \leqslant c\sigma$ *with some fixed* $c > 0$*, if and only if* $f = g$

Example. Let $0 < \alpha, \beta < 1$ be two algebraic irrationals. Let f, g be two additive functions defined by letting $f(p^k) = \{\alpha p\}$ and $g(p^k) = \{\beta p\}$ at all primes powers p^k. By Vinogradov's theorem (on the uniform distribution of $\{\alpha p\}$, see [19], ch. 11), both $f, g \in \mathcal{C}$ and in fact $\Psi(f;t) = t = \Psi(g;t)$ for $0 \leqslant t \leqslant 1$. Thus by Theorem 1.1, f, g are similarly distributed (i.e (1.6) holds) when Δ is in the range $1 \leqslant \Delta \leqslant o(\sigma)$ but not when $\Delta \asymp \sigma$ unless $f = g$, that is $\alpha = \beta$.

The theorem highlights a certain "discrete" behaviour of additive functions belonging to $\mathcal{C}$: For example, if (1.6) holds uniformly in $1 \leqslant \Delta \leqslant o(\sigma^{1/3+\varepsilon})$ with any fixed $\varepsilon > 0$, then (1.6) holds uniformly in $1 \leqslant \Delta \leqslant o(\sigma^{1/2})$. In fact, given any $\alpha \in (1/3;1)$, suppose that (1.6) holds uniformly in $1 \leqslant \Delta \leqslant o(\sigma^{\alpha})$, then for any $\delta > 0$ relation (1.6) holds in $1 \leqslant \Delta \leqslant o(\sigma^{\alpha+\delta})$ as long as $\rho(\alpha+\delta) = \rho(\alpha)$.

For a $f \in \mathcal{C}$ we have $\sigma^2(f;x) \sim c \log\log x$ for some constant $c > 0$. Thus given $f, g \in \mathcal{C}$ we can always find a constant $c > 0$ such that $\sigma(f;x) \sim \sigma(c \cdot g;x)$. Keeping this observation in mind and applying Theorem 1.1, we obtain the following corollary.

Corollary. *Let* $f, g \in \mathcal{C}$. *Suppose that* (1.6) *holds uniformly in* $1 \leqslant \Delta \ll \sigma \asymp (\log\log x)^{1/2}$, *then there is a constant* $c > 0$ *such that* $f = c \cdot g$.

(We mention another consequence of theorem 1.1 at the end of the introduction).

In the second part of the paper we focus on strongly additive f such that $0 \leqslant f(p) \leqslant O(1)$ and $\sigma(f;x) \to \infty$. We investigate the relationship between the asymptotic behaviour of

$$\mathcal{D}_f(x;\Delta) \;:= \frac{1}{x} \cdot \# \left\{ n \leqslant x : \frac{f(n) - \mu(f;x)}{\sigma(f;x)} \geqslant \Delta \right\} \tag{1.7}$$

in the range $1 \leqslant \Delta \ll_{\varepsilon} \sigma(f;x)^{1-\varepsilon}$ and the convergence properties of

$$\frac{1}{\sigma^2(f;x)} \sum_{\substack{p \leqslant x \\ f(p) \leqslant t}} \frac{f(p)^2}{p} \cdot \left(1 - \frac{1}{p}\right) \tag{1.8}$$

We prove roughly the following : If (1.8) converges to a distribution function $\Psi(\cdot)$ sufficiently fast, then $\mathcal{D}_f(x;\Delta)$ behaves asymptotically like a sum of $\sigma^2(f;x)$ independent and identically distributed random variables $X_1, X_2, \ldots$ with distribution determined by

$$\mathbb{E}\left[e^{it X_1}\right] = \exp\left(\int_{\mathbb{R}} \frac{e^{iut} - iut - 1}{u^2} d\Psi(u)\right). \tag{1.9}$$

Note that the above forces $\mathbb{E}[X_1] = 0$ and $\mathrm{Var}(X_1) = 1$. We do need to be more precise here since it is certainly possible that $\sigma^2(f;x)$ is not an integer: So, when we write "a sum of $\sigma^2(f;x)$ i.i.d random variables", we really mean a Levy process at time $t = \sigma^2(f;x)$, with initial distribution determined by (1.9).[2]

In the converse direction we prove that if (1.7) behaves asymptotically like a sum of $\sigma^2(f;x)$ i.i.d random variables (distributed according to (1.9) plus $\Psi(\alpha) - \Psi(0) = 1$ for some $\alpha > 0$) then (1.8) converges almost everywhere to the distribution function $\Psi(t)$.

The original motivation for studying this question was to characterize additive functions with a "Poisson distribution" on the integers. Namely, we wanted to show that any strongly additive functions whose values on the integers are "Poisson distributed" must be $1 + o(1)$ or $o(1)$ on most primes. Here is an example of what was achieved in this direction (the example is a particular case of the theorems discussed previously): For convenience denote by

$$\mathbb{P}\,\mathrm{oisson}\,(x;\Delta) = \sum_{k \geqslant x + \Delta\sqrt{x}} e^{-x} \cdot \frac{x^k}{k!}$$

the tails of a Poisson distribution with parameter $x \geqslant 0$. As a consequence [3] of a well-known result of Halász [10] if f is strongly additive, $f(p) \in \{0,1\}$ and $\sigma(f;x) \to \infty$, then

$$\mathcal{D}_f(x;\Delta) \sim \mathbb{P}\,\mathrm{oisson}\left(\sigma^2(f;x);\Delta\right) \tag{1.10}$$

uniformly in $1 \leqslant \Delta \leqslant o(\sigma(f;x))$. Conversely, suppose that f is strongly additive $0 \leqslant f(p) \leqslant O(1)$, $\sigma(f;x) \to \infty$ and that (1.10) holds uniformly in $1 \leqslant \Delta \leqslant o(\sigma(f;x))$. Then

$$\frac{1}{\sigma^2(f;x)} \sum_{\substack{p \leqslant x \\ f(p) \leqslant t}} \frac{f(p)^2}{p} \cdot \left(1 - \frac{1}{p}\right) \longrightarrow \delta(t) := \begin{cases} 1 \text{ if } t \geqslant 1 \\ 0 \text{ otherwise} \end{cases}$$

[2] Levy processes are defined below; for now we can think of it as a natural way to make continuous a count that is naturally discrete.

[3] Halász originally proved that $\mathcal{D}_f(x;\Delta) \sim \mathbb{P}\,\mathrm{oisson}(\mu(f;x);\Delta)$. This does not contradict our statement (1.10) because when $f(p) \in \{0;1\}$ we have $\mu(f;x) = \sigma^2(f;x) + O(1)$ and in particular $\mathbb{P}\,\mathrm{oisson}(\mu(f;x);\Delta) \sim \mathbb{P}\,\mathrm{oisson}(\sigma^2(f;x);\Delta)$ in the range $1 \leqslant \Delta \leqslant o(\sigma(f;x))$.

at all $t \in \mathbb{R}$, with the possible exception of $t = 1$. Therefore, for most primes p we either have $f(p) = 1 + o(1)$ or $f(p) = o(1)$, thus complementing Halász's result.

Let us mention, without giving a proof, one consequence of the above result. Suppose that $f \geqslant 0$ is bounded on the primes, $\sigma(f;x) \longrightarrow \infty$ and $\sigma^2(f;x) = c \log\log x + O(1)$ for some constant $c > 0$ (the last assumption certainly holds if $f \in \mathcal{C}$). The following holds: If $\mathcal{D}_f(x;\Delta) \sim \mathbb{P}\text{oisson}(\log\log x;\Delta)$ uniformly in $1 \leqslant \Delta \leqslant o(\sigma(f;x))$ then $f(p) = \sqrt{c} + o(1)$ for all but $o(\pi(x))$ primes $\leqslant x$ [4]. For a $f \in \mathcal{C}$ a more precise result follows from theorem 1.1 (and the fact that $\mathcal{D}_\omega(x;\Delta) \sim \mathbb{P}\text{oisson}(\log\log x;\Delta)$ uniformly in $1 \leqslant \Delta \leqslant o(\sigma(f;x))$ where $\omega(n)$ is the number of prime factors of n).

2. Precise Statement of results.

Building on earlier work by Kubilius ([12], p. 160) and Maciulis [14] we establish Theorem 2.1.

Theorem 2.1. *If* $f \in \mathcal{C}$ *and* $\sigma = \sigma(f;x)$ *then*

$$\mathcal{D}_f(x;\Delta) \sim \mathbb{P}\left(\sum_{p \leqslant x} f(p)\left[X_p - \frac{1}{p}\right] \geqslant \Delta\sigma\right) \tag{2.1}$$

uniformly in $1 \leqslant \Delta \leqslant o(\sigma)$.

Remark. One can prove (2.1) for many other classes of f (for example when $f \geqslant 0$ is a strongly additive function such that $\sigma(f;x) \to \infty$ and $0 \leqslant f(p) \leqslant O(1)$). We believe that (2.1) holds in very broad generality, perhaps even for any f satisfying the Erdös-Kac theorem (in the Kubilius-Shapiro version, see [5], theorem 12.2), though one may have to introduce some natural restrictions.

As announced in the introduction, the asymptotic relation (2.1) fails when $\Delta \asymp \sigma(f;x)$. This phenomenon is described in the next Theorem.

Theorem 2.2. *Let* $f \in \mathcal{C}$. *Let* $\sigma = \sigma(f;x)$. *For fixed* $\delta > 0$, *uniformly in* $1 \leqslant \Delta \leqslant \delta\sigma$,

$$\mathcal{D}_f(x;\Delta) \sim \mathcal{A}\left(f;\frac{\Delta}{\sigma}\right) \cdot \mathbb{P}\left(\sum_{p \leqslant x} f(p) \cdot \left[X_p - \frac{1}{p}\right] \geqslant \Delta\sigma\right)$$

The function $\mathcal{A}(f;x)$ *is analytic (in a neighborhood of* $\mathbb{R}^+ \cup \{0\}$*), strictly decreasing, and decays to* 0 *as* $x \to \infty$. *Further* $\mathcal{A}(f;0) = 1$. *All those properties are the consequence of an explicit formula for* $\mathcal{A}(f;z)$ *that we now describe. Denote by*

$$\widehat{\Psi}(f;z) = \int_{\mathbb{R}} e^{zt} d\Psi(f;t)$$

[4] If instead we assume $\sigma^2(f;x) = c\mu(f;x) + O(1)$ and $\mathcal{D}_f(x;\Delta) \sim \mathbb{P}\text{oisson}(\mu(f;x);\Delta)$ then necessarily $c = 1$ and $f(p) = 1 + o(1)$ or $f(p) = o(1)$ for most primes p.

the Laplace transform of the distribution function $\Psi(f;t)$. *Let* $\omega(z) = \omega(f;z)$ *be defined implicitly by* $\widehat{\Psi}'(f;\omega(z)) = \widehat{\Psi}'(f;0) + z\widehat{\Psi}''(f;0)$. *The function* $\omega(f;z)$ *thus defined is well-defined in a neighborhood of* $\mathbb{R}^+ \cup \{0\}$ *and analytic there. We have*

$$\mathcal{A}(f;z) = \frac{e^{-\gamma(\widehat{\Psi}(f;\omega(z))-1)}}{\Gamma(\widehat{\Psi}(f;\omega(z))}$$

Example. In the case of f being the number of prime factors of n, we find that $\widehat{\Psi}(f;z) = e^z$ and that $\omega(f;z) = \log(1+z)$. Therefore

$$\mathcal{A}(f;z) = \frac{e^{-\gamma z}}{\Gamma(1+z)}$$

Thus $\mathcal{A}(f;x)$ decays very fast !

The function $\mathcal{A}(f;\Delta/\sigma)$ stays essentially constant throughout the range $\Delta = c\sigma + o(\sigma)$ (where $\sigma = \sigma(f;x)$), since $A(f;\Delta/\sigma) = A(f;c) + o(1)$ by analyticity. In this respect when $\Delta \sim c\sigma$ the quantity $\mathcal{D}_f(x;\Delta)$ differs asymptotically from its probabilistic counterpart only by a constant.

We believe that the appearance of the function $\mathcal{A}(f;z)$ is essentially due to the large prime factors. To back up our claim, let us look at what happens when one ignores the large prime factors. We denote by $f(n;y)$ the truncated additive function

$$f(n;y) = \sum_{\substack{p|n \\ p \leqslant y}} f(p)$$

The following conjecture was suggested by Kevin Ford.

Conjecture 2.3. *Suppose that* $u := \log x/\log y \longrightarrow \infty$ *and* $u \leqslant \operatorname{loglog} x$. *Then*

$$\frac{1}{x} \cdot \#\left\{n \leqslant x : \frac{f(n;y) - \mu(f;y)}{\sigma(f;y)} \geqslant \Delta\right\} \sim \mathbb{P}\left(\sum_{p \leqslant y} f(p)\left[X_p - \frac{1}{p}\right] \geqslant \Delta\sigma(f;y)\right)$$

uniformly in $1 \leqslant \Delta \leqslant c\sigma(f;y)$ *for any fixed* $c > 0$.

In support of the conjecture we have the following simple proposition (which we deduce from Kubilius's theorem, in Barban-Vinogradov's version, [4], lemma 3.2, p. 122).

Proposition 2.4. *Suppose that* $u = \log x/\log y \asymp \operatorname{loglog} x$. *Then*

$$\frac{1}{x} \cdot \#\left\{n \leqslant x : \frac{f(n;y) - \mu(f;y)}{\sigma(f;y)} \geqslant \Delta\right\} \sim \mathbb{P}\left(\sum_{p \leqslant y} f(p)\left[X_p - \frac{1}{p}\right] \geqslant \Delta\sigma(f;y)\right)$$

uniformly in $1 \leqslant \Delta \leqslant c\sigma(f;y)$ *for any fixed* $c > 0$.

Theorem 2.2 seems to suggest the inequality

$$\mathcal{D}_f(x;\Delta) \quad \leqslant \quad (1+o(1))\cdot\mathbb{P}\left(\sum_{p\leqslant x} f(p)\left[X_p-\frac{1}{p}\right]\geqslant \Delta\sigma(f;x)\right) \tag{2.2}$$

might be true in general. This is possibly true for fixed strongly additive $f \geqslant 0$, uniformly in $\Delta \geqslant 0$. It is certainly false if we drop the condition $f \geqslant 0$ and allow both f and Δ to vary uniformly. Ruzsa's paper (see [16]) contains a weaker version of (2.2) which is however valid uniformly in f and Δ. There is also a discussion of the "optimal" inequality in Tenenbaum's book (see [17], p. 315).

We now turn to the following question : Given an additive function f what is the relationship between the distribution of f on the primes and the distribution of f on the integers ? An early result in that direction is Kubilius's theorem, stated below (see [5], p. 12).

Theorem. *Let f be an additive function. Let $\sigma = \sigma(f;x)$. Suppose that for every fixed $t \in (0;1)$ we have $\sigma(f;x) - \sigma(f;x^t) = o(\sigma(f;x))$. The following equivalence holds: There is a distribution function $\Psi(\cdot)$ such that*

$$\frac{1}{\sigma^2}\sum_{\substack{p\,\leqslant\, x\\ f(p)\,\leqslant\, t\sigma}}\frac{f(p)^2}{p}\cdot\left(1-\frac{1}{p}\right)$$

converges weakly to $\Psi(t)$ if and only if, there is a distribution function F with mean 0 and variance 1 such that $\mathcal{D}_f(x;t)$ converges weakly to $1-F(t)$. The relationship between Ψ and F is determined by

$$\int_{\mathbb{R}} e^{iut}\,dF(t) = \exp\left(\int_{\mathbb{R}}\frac{e^{iut}-iut-1}{u^2}d\Psi(u)\right).$$

The striking feature of Kubilius's theorem is that from the statistical behaviour of $f(\cdot)$ on the integers one is able to deduce the statistical behaviour of $f(\cdot)$ on the primes. The simplest case in which the theorem is applicable, is when f is equal to the number of prime factors of n. In this case

$$\frac{1}{\sigma^2(f;x)}\sum_{\substack{p\,\leqslant\, x\\ f(p)\,\leqslant\, t\sigma}}\frac{f(p)^2}{p} \quad\longrightarrow\quad \delta_0(t) := \begin{cases} 1 \text{ if } t>0\\ 0 \text{ if } t\leqslant 0\end{cases} \tag{2.3}$$

because all the $f(p)$'s belong to a bounded range. As a consequence the limit law $1-F(t) = \lim \mathcal{D}_f(x;t)$ is normal. This is of course nothing else than a variation on the Erdös-Kac theorem. However (2.3) holds, in fact, for all $f \in \mathcal{C}$. Thus, all that Kubilius's theorem is saying about additive function $f \in \mathcal{C}$ is that the limit law $1-F(t) = \lim \mathcal{D}_f(x;t)$ is normal. In what follows we will be interested in obtaining more detailed information about the convergence of

$$\frac{1}{\sigma^2(f;x)} \sum_{\substack{p \leqslant x \\ f(p) \leqslant t}} \frac{f(p)^2}{p} \cdot \left(1 - \frac{1}{p}\right) \longrightarrow \Psi(t) \text{ a.e} \tag{2.4}$$

from assumptions on the large deviation behaviour of $\mathcal{D}_f(x;\Delta)$ when $1 \leqslant \Delta \ll_\varepsilon \sigma^{1-\varepsilon}$ and vice-versa. Note the difference in scale between (2.3) and (2.4) (that is, $t\sigma$ is replaced by t). To state the next two results let us define a *Levy process* (or rather a diluted version of that notion: we don't need any assumptions on the underlying probability space - the name *Levy process in law* would seem appropriate but it is already taken). We allow ourselves a little sloppiness in the definition (the sloppiness comes from working with an uncountable set of mutually independent random variables, without discussing the existence of such a family).

Definition. *Let Ψ be a distribution function. Denote by $\{\mathcal{Z}_\Psi(u) : u > 0\}$ an indexed family of mutually independent random variables, with distribution determined by*

$$\mathbb{E}\left[e^{it\mathcal{Z}_\Psi(u)}\right] = \exp\left(u \cdot \int_{\mathbb{R}} \frac{e^{itx} - itx - 1}{x^2} d\Psi(x)\right) \tag{2.5}$$

Note that for each $u > 0$ the random variable $\mathcal{Z}_\Psi(u)$ has mean 0 and variance u.

It is clear that the distribution of $\mathcal{Z}_\Psi(u)$ is known once the distribution of $\mathcal{Z}_\Psi(1)$ is. Also, when n is a positive integers we can write $\mathcal{Z}_\Psi(n) \overset{\text{law}}{=} X_1 + \ldots + X_n$ with $X_1, X_2, \ldots$ independent and identically distributed random variables, each being distributed in exactly the same way as $\mathcal{Z}_\Psi(1)$. Thus $\mathcal{Z}_\Psi(x)$ is a rather natural "continuous" generalization of the notion of a "sum of n independent and identically distributed random variables". Finally, let us note that in the special case when $\Psi(t)$ has a jump of size 1 at $t = 1$, the random variable $\mathcal{Z}_\Psi(x)$ is a centered Poisson random variable with parameter x.

The content of the next Theorem is that a nice distribution on the primes implies a nice distribution on the integers. Following Elliott (see [6], p. 50) we consider this a Theorem in the "primes to integers" direction.

Theorem 2.5. *Let f be a strongly additive function. Suppose that $\sigma^2 = \sigma^2(f;x) \to \infty$ and that $0 \leqslant f(p) \leqslant O(1)$ for all primes p. If there is a distribution function $\Psi(t)$ such that*

$$\frac{1}{\sigma^2} \sum_{\substack{p \leqslant x \\ f(p) \leqslant t}} \frac{f(p)^2}{p} \cdot \left(1 - \frac{1}{p}\right) - \Psi(t) \ll \frac{1}{\sigma^2}$$

uniformly in $t \in \mathbb{R}$ as $x \longrightarrow \infty$, then

$$\mathcal{D}_f(x;\Delta) \sim \mathbb{P}\left(\mathcal{Z}_\Psi\left(\sigma^2\right) \geqslant \Delta\sigma\right)$$

uniformly in $1 \leqslant \Delta \leqslant o(\sigma)$, as $x \to \infty$.

Theorem 2.5 is saying that assuming certain regularity conditions on the primes, the distribution of an additive function on the integers mimics a sum of $\sigma^2(f;x)$ random variables. In the converse "integers to primes" direction we have Theorem 2.6.

Theorem 2.6. *Let* f *be a strongly additive function. Suppose that* $\sigma^2 = \sigma^2(f;x) \longrightarrow \infty$ *and that* $0 \leqslant f(p) \leqslant O(1)$ *for all primes* p*. If we have*

$$\mathcal{D}_f(x;\Delta) \sim \mathbb{P}\left(\mathcal{Z}_\Psi\left(\sigma^2\right) \geqslant \Delta\sigma\right)$$

uniformly in $1 \leqslant \Delta \ll_\varepsilon \sigma^{1-\varepsilon}$*, for some distribution function* Ψ *of compact support on* $\mathbb{R}_{\geqslant 0}$ *(that is* $\Psi(\alpha) - \Psi(0) = 1$ *for some* $\alpha > 0$*), then*

$$\frac{1}{\sigma^2(f;x)} \sum_{\substack{p \leqslant x \\ f(p) \leqslant t}} \frac{f(p)^2}{p} \cdot \left(1 - \frac{1}{p}\right)$$

converges weakly to the distribution function Ψ *(i.e converges to* $\Psi(t)$ *at all continuity points of* t*).*

(This is not an "integer to primes" theorem in the sense of [6], because of the $\sigma^2(f;x) \to \infty$ and $f(p) \leqslant O(1)$ assumption; we believe both can be dropped without (too) much difficulty). The motivation for Theorem 2.6 and Theorem 2.5 comes from an open-ended question raised in Elliott's book [6]: given the "average" behaviour of an additive function f on the integers, how much can we say about its behaviour on the primes? (see p. 50 in [6]).

By taking $\Psi(t)$ to have a jump of size 1 at $t = 1$ in the previous theorem, we obtain the following corollary.

Corollary 2.7. *Let*

$$\mathbb{P}\,\mathrm{oisson}\,(x;\Delta) = \sum_{k \geqslant x + \Delta\sqrt{x}} e^{-x} \cdot \frac{x^k}{k!}$$

denote the tails of a Poisson distribution with parameter x*. By a result of Halász* [10] *for any strongly additive function* f *such that* $f(p) \in \{0,1\}$ *and* $\sigma^2(f;x) \to \infty$*, we have* $\mathcal{D}_f(x;\Delta) \sim \mathbb{P}\,\mathrm{oisson}(\sigma^2(f;x);\Delta)$ *uniformly in the range* $1 \leqslant \Delta \leqslant o(\sigma(f;x))$*. Conversely, given a strongly additive function* f *such that* $0 \leqslant f(p) \leqslant O(1)$ *and* $\sigma^2(f;x) \to \infty$*, suppose that* $\mathcal{D}_f(x;\Delta) \sim \mathbb{P}\,\mathrm{oisson}(\sigma^2(f;x);\Delta)$ *holds uniformly in the range* $1 \leqslant \Delta \ll_\varepsilon \sigma(f;x)^{1-\varepsilon}$*; then*

$$\frac{1}{\sigma^2(f;x)} \sum_{\substack{p \leqslant x \\ f(p) \leqslant t}} \frac{f(p)^2}{p} \cdot \left(1 - \frac{1}{p}\right) \longrightarrow \delta(t) := \begin{cases} 1 \text{ if } t \geqslant 1 \\ 0 \text{ else} \end{cases}$$

at all $t \in \mathbb{R}$*, except possibly at* $t = 1$*. Thus for almost all primes* p *we either have* $f(p) = 1 + o(1)$ *or* $f(p) = o(1)$*.*

We now turn to the description of the technical Theorem 2.8 which we state in the introduction because of its central importance. A recurrent difficulty in the paper is that when we deal with the distribution of f in the range $\Delta \asymp \sigma$ we are forced to consider the following two cases separately:

1. The values $f(p)$ do cluster on $\alpha\mathbb{Z}$ for some $\alpha > 0$.
2. The values $f(p)$ do not cluster on $\alpha\mathbb{Z}$ for any $\alpha > 0$.

Definition. *Let X be a random variable. We say that X is lattice distributed on $\alpha\mathbb{Z}$ ($\alpha > 0$) if $\mathbb{P}(X \in \alpha\mathbb{Z}) = 1$, and $\mathbb{P}(X \in \beta\mathbb{Z}) < 1$ for all $\beta > \alpha$. Analogously we say that a distribution function is lattice distributed (resp. non-lattice distributed) when the underlying random variable is lattice (resp. non-lattice distributed).*

In order to state Theorem 2.8 we introduce further notation. We denote by

$$\widehat{\Psi}(f;z) := \int_{\mathbb{R}} e^{zt} d\Psi(f;t) = 1 + \sum_{k=1}^{\infty} \int_{\mathbb{R}} t^k d\Psi(f;t) \cdot \frac{z^k}{k!}, \tag{2.6}$$

the two-sided Laplace transform of $\Psi(f;t)$. By (1.3) and (1.4) we have $1 - \Psi(f;t) \ll_A e^{-At}$ and $\Psi(f;t) = 0$ for $t < 0$. Hence the Laplace transform $\widehat{\Psi}$ is an entire function with Taylor expansion as in (2.6). Thus all moments of $\Psi(f;t)$ exists, and in accordance with (2.6) the k-th moment of $\Psi(f;\cdot)$ is$\int_{\mathbb{R}} t^k d\Psi(f;t)$. We also define the function $\omega(f;z)$ implicitly by,

$$\widehat{\Psi}'(f;\omega(f;z)) = \widehat{\Psi}'(f;0) + z \cdot \widehat{\Psi}''(f;0)$$

Although this function does not appear in the statement of Theorem 2.8, it will frequently be encountered in subsequent proofs.

Theorem 2.8. *Let $f \in \mathcal{C}$. We have,*

(1) Uniformly in $1 \leqslant \Delta \leqslant o\left(\sigma(f;x)^{1/3}\right)$,

$$\mathcal{D}_f(x;\Delta) \sim \frac{1}{\sqrt{2\pi}} \int_{\Delta}^{\infty} e^{-u^2/2} \cdot du$$

(2) Given $\varepsilon > 0$, uniformly in $(\log\log x)^{\varepsilon} \ll \Delta \leqslant o(\sigma(f;x))$,

$$\mathcal{D}_f(x;\Delta) \sim S_f(x;\Delta) := \frac{(\log x)^{\widehat{\Psi}(f;\nu)-1-\nu\widehat{\Psi}'(f;\nu)}}{\nu(2\pi\widehat{\Psi}''(f;\nu)\log\log x)^{1/2}}$$

Here $\nu = \nu_f(x;\Delta)$ is a parameter, defined as the unique positive solution to the equation

$$\widehat{\Psi}'(f;\nu)\log\log x = \widehat{\Psi}'(f;0)\log\log x + \Delta \cdot (\widehat{\Psi}''(f;0)\log\log x)^{1/2}$$

(3) If $\Psi(f;t)$ is not lattice distributed, then given $\delta, \varepsilon > 0$, uniformly in the range $(\log\log x)^{\varepsilon} \ll \Delta \leqslant \delta\sigma(f;x)$,

$$\mathcal{D}_f(x;\Delta) \sim \frac{L(f;\nu)e^{-\nu c(f)}}{\Gamma(\widehat{\Psi}(f;\nu))} \cdot S_f(x;\Delta)\ ,\ \nu = \nu_f(x;\Delta)$$

where $L(f;z)$ *is an (entire) function, defined by*

$$L(f;z) \;=\; \prod_p \left(1-\frac{1}{p}\right)^{\widehat{\Psi}(f;z)} \cdot \left(1+\frac{e^{zf(p)}}{p-1}\right)$$

and $c(f)$ *is defined by* $\mu(f;x) = \widehat{\Psi}'(f;0)\cdot \log\log x + c(f) + o(1)$.

(4) If $\Psi(f;t)$ *is lattice distributed on* $\mathbb{Z}$*, then given* $\delta, \varepsilon > 0$*, uniformly in the range* $(\log\log x)^{\varepsilon} \ll \Delta \leqslant \delta\sigma(f;x)$*,*

$$\mathcal{D}_f(x;\Delta) \sim \frac{L(\mathfrak{g};\nu)e^{-\nu c(f)}}{\Gamma(\widehat{\Psi}(f;\nu))} \cdot \mathcal{P}_{\mathfrak{h}}\left(\xi_f(x;\Delta);\nu\right) \cdot S_f(x;\Delta)\ ,\ \nu = \nu_f(x;\Delta)$$

where $\xi_f(x;\Delta) = \mu(f;x) + \Delta\sigma(f;x)$*, and* $\mathfrak{g}, \mathfrak{h}$ *are two additive functions defined by*

$$\mathfrak{g}(p) = \begin{cases} f(p) \text{ if } f(p) \in \mathbb{Z} \\ 0 \quad \text{otherwise} \end{cases} \quad \text{and} \quad \mathfrak{h}(p) = \begin{cases} f(p) \text{ if } f(p) \notin \mathbb{Z} \\ 0 \quad \text{otherwise} \end{cases}$$

Finally, the function $\mathcal{P}_{\mathfrak{h}}(a;\nu)$ *is defined by,*

$$\mathcal{P}_{\mathfrak{h}}\left(a;\nu\right) = \nu \sum_{\ell\in\mathbb{Z}} e^{\nu(\ell+\{a\})} \cdot \mathbb{P}\left(\sum_p \mathfrak{h}(p)X_p \geqslant \ell + \{a\}\right)$$

Remark. In part (4) of Theorem 2.8 the assumption "$\Psi(f;t)$ lattice distributed on $\mathbb{Z}$" entails no loss of generality. If $\Psi(f;t)$ is lattice distributed on $\alpha\mathbb{Z}$ then $\Psi(f/\alpha;t) = \Psi(f;\alpha t)$ is lattice distributed on $\mathbb{Z}$ and f/α is an additive function.

Let us make a few remarks about the asymptotics in Theorem 2.8. In the range $1 \leqslant \Delta \leqslant o(\sigma)$ the parameter $\nu := \nu_f(x;\Delta)$ is $o(1)$. In addition ν admits an convergent expansion of the form $\sum_k a_k(\Delta/\sigma)^k$ and so does the function $A(f;z) = \sum_k b_k z^k$. On composing the two we obtain $(\log x)^{A(f;\nu)} = \exp(\log\log x \sum c_k(\Delta/\sigma)^k) \sim \exp(\sigma^2 \sum c_k(\Delta/\sigma)^k)$ for some coefficients c_k. Thus for $\Delta \leqslant \sigma^{1-\varepsilon}$ only the first $\ll 1/\varepsilon$ terms will dictate the asymptotic behaviour of $(\log x)^{A(f;\nu)}$. To complete the picture,

$$\frac{e^{\Delta^2/2}}{\sqrt{2\pi}} \int_{\Delta}^{\infty} e^{-u^2/2}\,du \sim \frac{1}{\sqrt{2\pi}\Delta} \text{ for } \Delta \to \infty$$

On the other hand when $\Delta \sim c\sigma$ for some fixed constant $c > 0$, all the c_k have a non-trivial contribution and the the parameter $\nu = \nu_f(x;\Delta) = \kappa + o(1)$ for some $\kappa > 0$ depending on c. In the range $\Delta \sim c\sigma$ both $L(f;\nu)$ and $e^{-\nu c(f)}$ are essentially constant, while $S_f(x;\Delta)$ is about the size of $(\log x)^{\widehat{\Psi}(f;\kappa)-1-\kappa\widehat{\Psi}'(f;\kappa)+o(1)}$. Hopefully, these few remarks give a good picture of the asymptotic behaviour in part (1), (2) and (3) of theorem 2.8. Regarding part (4) of theorem 2.8, since the $f(p)$ are concentrated on $\mathbb{Z}$, we have $\mathfrak{g}(p) = f(p)$ for "almost all" prime p. Throughout the range $\Delta \asymp \sigma$ we have $L(\mathfrak{g};\nu) \asymp 1$ and $\mathcal{P}_{\mathfrak{h}}(\xi_f(x;\Delta);\nu) \asymp 1$ but otherwise the latter expression is highly irregular. In fact, because $\mathcal{P}_{\mathfrak{h}}(\xi_f(x;\Delta);\nu)$ involves $\{\xi_f(x;\Delta)\} = \{\mu(f;x) + \Delta\sigma(f;x)\}$ the ratio

$$\mathcal{D}_f(x;c\sigma(f;x)) \cdot S_f(x;c\sigma(f;x))^{-1}$$

does not tend to a limit as $x \to \infty$, when c is fixed. This is also discussed in Balazard, Nicolas, Pomerance, and Tenenbaum's paper [1]. Let us note in passing that the probabilistic interpretation we give for $\mathcal{P}_{\mathfrak{h}}(\xi_f(x;\Delta);\nu)$ might be of interest in connection with some of the question raised in [3]. Compared to previous results, namely those of the Lithuanian school, the novelty in Theorem 2.8 is the bigger range $1 \leqslant \Delta \leqslant \delta\sigma(f;x)$, although it is quite possible that the result was known, or at least anticipated, by the experts in the field.

Plan of the paper. Section 4-8 and Section 9-10 can be taught of as separate. In sections 4.3-4.5 we establish rather general large deviations results. The lemmas in section 4.1 will allow to specialize these to cases of arithmetical interest. In section 5, we deduce theorem 2.8 from the lemmas in section 4. In section 6 we establish theorem 1.1 by using theorem 2.8. Finally we prove theorem 2.2 in section 7. Theorems 2.6 and 2.7 are proven respectively in section 10 and section 9.

Regarding sections 4-8, the core ideas are scattered throughout the proof of proposition 4.10, the proof of proposition 4.17 and the entire section 6. Sections 9 and 10 are essentially "stand-alone" and the techniques used there differ from the ones appearing before.

3. Notation

We summarize in the table below some of the recurrent notation. We let $f \in \mathcal{C}$.

$$
\begin{aligned}
\mathcal{D}_f(x;\Delta) &:= \frac{1}{x} \cdot \# \left\{ n \leqslant x : \frac{f(n) - \mu(f;x)}{\sigma(f;x)} \geqslant \Delta \right\} \\
L(f;z) &:= \prod_p \left(1 - \frac{1}{p}\right)^{\widehat{\Psi}(f;z)} \cdot \left(1 + \frac{e^{zf(p)}}{p-1}\right) \\
\omega(f;z) &:= \text{defined implicitly by } \widehat{\Psi}'(f;\omega(f;z)) = \widehat{\Psi}'(f;0) + z \cdot \widehat{\Psi}''(f;0) \\
\sigma_\Psi^2(f;x) &:= \widehat{\Psi}''(f;0) \cdot \operatorname{loglog} x \ = \int_{\mathbb{R}} t^2 d\Psi(f;t) \cdot \operatorname{loglog} x \\
c(f) &:= \mu(f;x) - \widehat{\Psi}'(f;0) \cdot \operatorname{loglog} x + o(1) \\
\nu_f(x;\Delta) &:= \omega(f;\Delta/\sigma_\Psi(f;x)) \\
A(f;z) &:= \widehat{\Psi}(f;z) - 1 - z\widehat{\Psi}'(f;z) \\
\mathcal{E}(f;z) &:= A(f;\omega(f;z)) \\
S_f(x;\Delta) &:= \frac{(\log x)^{\widehat{\Psi}(f;\nu) - \nu\widehat{\Psi}'(f;\nu) - 1}}{\nu(2\pi\widehat{\Psi}''(f;\nu)\operatorname{loglog} x)^{1/2}} \text{ with } \nu = \nu_f(x;\Delta) \\
\mathfrak{h}_f(n) &:= \text{strongly additive function such that } \mathfrak{h}_f(p) = \begin{cases} f(p) \text{ if } f(p) \notin \mathbb{Z} \\ 0 \quad \text{otherwise} \end{cases} \\
S(\mathfrak{h}) &:= \{p : \mathfrak{h}(p) \neq 0\} = \{p : f(p) \notin \mathbb{Z}\} \\
\mathfrak{g}_f(n) &:= f(n) - \mathfrak{h}_f(n) \\
\{X_p\} &:= \text{Independent Bernoulli random variable with } \mathbb{P}(X_p = 1) = 1/p. \\
X(\mathfrak{h}_f) &:= \sum_{p \leqslant x} \mathfrak{h}_f(p) X_p \\
\mathcal{P}_{\mathfrak{h}_f}(a;\nu) &:= \nu \sum_{k \in \mathbb{Z}} e^{\nu(k + \{a\})} \cdot \mathbb{P}(X(\mathfrak{h}_f) \geqslant k + \{a\}) \\
\xi_f(x;\Delta) &:= \mu(f;x) + \Delta\sigma(f;x) \\
B^2(f;x) &:= \sum_{p \leqslant x} \frac{f(p)^2}{p} \\
\mathcal{D}_f^\times(x;\Delta) &:= \frac{1}{x} \cdot \# \left\{ n \leqslant x : \frac{f(n) - \mu(f;x)}{B(f;x)} \geqslant \Delta \right\}
\end{aligned}
$$

Sometimes we will write $\log_k x$ to mean the k times iterated logarithm. When the context is clear we will drop the subscript f from $\mathfrak{h}_f$ and $\mathfrak{g}_f$. In the same vein we usually abbreviate $\nu_f(x;\Delta)$ by ν, and sometimes $\sigma_\Psi(f;x)$ by σ_Ψ, although this is always mentioned when done.

4. Preliminary lemmata

In the first two subsections we collect background information. The main technical tools are developed in the subsequent sections: indeed, the three general large deviations theorems corresponding to Proposition 4.9, Proposition 4.10 and Proposition 4.17, form the technical backbone of this paper.

4.1. **A mean-value theorem.**

The object of this section is to prove the following mean value theorem.

Proposition 4.1. *Let* $f \in \mathcal{C}$. *Given* $C > 0$, *uniformly in* $-C \leqslant \kappa := \operatorname{Re} s \leqslant C$,

$$\frac{1}{x}\sum_{n \leqslant x} e^{sf(n)} = \frac{L(f;s)}{\Gamma(\widehat{\Psi}(f;s))} \cdot (\log x)^{\widehat{\Psi}(f;s)-1} + O_{A,C}\left(\mathcal{E}_A(x;s) \cdot (\log x)^{\widehat{\Psi}(f;\kappa)-2}\right)$$

where $\mathcal{E}_A(x;s) = 1 + |\operatorname{Im} s|^{1/A} + |\operatorname{Im} s|/\log x$. *In particular, given* $C > 0$, *we have, uniformly in* $-C \leqslant \kappa := \operatorname{Re} s \leqslant C$ *and* $|\operatorname{Im} s| \leqslant \log x$,

$$\frac{1}{x}\sum_{n \leqslant x} e^{sf(n)} = \frac{L(f;s)}{\Gamma(\widehat{\Psi}(f;s))} \cdot (\log x)^{\widehat{\Psi}(f;s)-1} + O_C\left((\log x)^{\widehat{\Psi}(f;\kappa)-3/2}\right)$$

Remark. We will be mostly using the second formula.

With more effort one can (probably) show that for $|\operatorname{Im} s| \leqslant \log x$ the error term is $(\log x)^{\operatorname{Re}(\widehat{\Psi}(f;s))-2}$, but this will not be needed. The lemma is proven by using the method of Levin and Fainleb (see [7] for a survey article and [13] for the paper we will follow). We include the proof only for completeness's sake. It is quite likely that a comparable result can be deduced directly from one of the lemma in Tenenbaum's book [17] but maybe only for a more restrained class of additive functions.

First let us prove that $\widehat{\Psi}(f;z)$ is entire.

Lemma 4.2. *Let* $f \in \mathcal{C}$. *The function* $\widehat{\Psi}(f;s)$ *is entire.*

Proof. Since $f \in \mathcal{C}$, by assumption (1.3) and (1.4)

$$1 - \Psi(f;t) \leqslant c(A) \cdot e^{-At}$$

for every fixed $A > 0$ and $c(A)$ a constant depending on A. Since in addition we require f to be positive, $\Psi(f;t) = 0$ when $t < 0$. It follows that

$$\begin{aligned}
\int_{\mathbb{R}} t^k d\Psi(f;t) &= \int_0^\infty t^k d\Psi(f;t) \\
&= k\int_0^\infty t^{k-1} \cdot (1 - \Psi(f;t))dt \\
&\leqslant c(A) \cdot k\int_0^\infty t^{k-1}e^{-At}dt \;=\; c(A) \cdot k \cdot k! \cdot A^{-k}
\end{aligned}$$

Therefore the series

$$1 + \sum_{k \geqslant 1} \int_{\mathbb{R}} t^k d\Psi(f;t) \cdot \frac{s^k}{k!}$$

converges absolutely in $|s| < A/2$. This allows us to interchange summation and integration and we obtain

$$1 + \sum_{k \geqslant 1} \int_{\mathbb{R}} t^k d\Psi(f;t) \cdot \frac{s^k}{k!} \;=\; \int_{\mathbb{R}} e^{st} d\Psi(f;t) \;:=\; \widehat{\Psi}(f;s)$$

for $|s| < A/2$. Since the series on the left is absolutely convergent for $|s| < A/2$ and sums to $\widehat{\Psi}(f;s)$ it follows that $\widehat{\Psi}(f;s)$ is analytic in $|s| < A/2$. But A is arbitrary, therefore $\widehat{\Psi}(f;s)$ is entire. □

Lemma 4.3. *Let $f \in \mathcal{C}$. Given $A, C > 0$ we have, uniformly in $|\operatorname{Re} s| \leqslant C$,*

$$\sum_{p \leqslant x} e^{sf(p)} = \pi(x) \cdot \left[\widehat{\Psi}(f;s) + O_{A,C}\left(\frac{1 + |\operatorname{Im} s|}{(\log x)^{2A}} \right) \right]$$

In particular the estimate

$$\sum_{p \leqslant x} e^{sf(p)} = \pi(x) \cdot \left[\widehat{\Psi}(f;s) + O_{A,C}\left((\log x)^{-A} \right) \right]$$

holds uniformly in $|\operatorname{Re} s| \leqslant C$ and $|\operatorname{Im} s| \leqslant (\log x)^A$ (hence also for $|\operatorname{Im} s| \leqslant \log\log x$).

Proof. To simplify notation let $F(x;t) = (1/\pi(x)) \cdot \#\{p \leqslant x : f(p) \leqslant t\}$. Let $A > 0$ be an arbitrary, but fixed constant, and write $\xi := \log\log x$. We have

$$\begin{aligned} \sum_{p \leqslant x} e^{sf(p)} &= \pi(x) \cdot \int_0^{\infty} e^{st} dF(x;t) \\ &= \pi(x) \cdot \left[\int_0^{A\xi} e^{st} dF(x;t) + \int_{A\xi}^{\infty} e^{st} dF(x;t) \right] \end{aligned} \tag{4.1}$$

Since $F(x;t)$ is a distribution function the second integral is for $\operatorname{Re} s \leqslant C$, bounded in modulus by $\int_{A\xi}^{\infty} e^{Ct} dF(x;t)$ which is $\ll (\log x)^{-2A}$ since $1 - F(x;t) \ll_C e^{-(C+2)t}$ by assumptions. We rewrite the first integral in (4.1) as

$$\int_0^{A\xi} e^{st} dF(x;t) = \int_0^{A\xi} e^{st} d\Psi(f;t) + \int_0^{A\xi} e^{st} d(F(x;t) - \Psi(f;t)) \tag{4.2}$$

Since $F(x;t) - \Psi(f;t) \ll_{C,A} (\log x)^{-A(C+3)}$ (again by assumptions) the second integral in (4.2) is bounded by $\ll (\log x)^{-2A} + |s| \cdot (\log x)^{-2A}$ which is less than $\ll (1 + |\operatorname{Im} s|) \cdot (\log x)^{-2A}$ because $|\operatorname{Re} s| \leqslant C$. As for the first integral in (4.2) we note that $1 - \Psi(f;t) \ll e^{-(C+2)t}$

hence $|\int_{A\xi}^{\infty} e^{st}d\Psi(f;t)| \leqslant \int_{A\xi}^{\infty} e^{Ct}d\Psi(f;t) \ll (\log x)^{-2A}$ which allows us to complete the tails. By (4.2) and the above observations

$$\int_0^{A\xi} e^{st}dF(x;t) = \int_0^{\infty} e^{st}d\Psi(f;t) + O\left(\frac{1+|\operatorname{Im} s|}{(\log x)^{2A}}\right) = \widehat{\Psi}(f;s) + O\left(\frac{1+|\operatorname{Im} s|}{(\log x)^{2A}}\right)$$

Plugging the above back into (4.1) and recalling that the second integral in (4.1) was bounded by $O((\log x)^{-2A})$ we conclude that

$$\sum_{p\leqslant x} e^{sf(p)} = \pi(x)\cdot\left[\widehat{\Psi}(f;s) + O_{A,C}\left(\frac{1+|\operatorname{Im} s|}{(\log x)^{2A}}\right)\right]$$

as desired. □

We now focus on $L(f;z)$. We prove that $L(f;z)$ is entire – this is used all over the place, but especially in the proof of the "structure theorem".

Lemma 4.4. *Let* $f \in \mathcal{C}$. *The function* $L(f;z)$ *is entire. Given* $\kappa > 0$ *there is an* $x_0(\kappa)$ *such that uniformly in* $|\operatorname{Re} z| \leqslant \kappa$, $|\operatorname{Im} z| \leqslant \log\log x$ *and* $x \geqslant x_0(\kappa)$,

$$\prod_{p\leqslant x}\left(1-\frac{1}{p}\right)^{\widehat{\Psi}(f;z)}\cdot\left(1+\frac{e^{zf(p)}}{p-1}\right) = L(f;z)\cdot\left(1+O\left((\log x)^{-1}\right)\right)$$

Furthermore, uniformly in $|\operatorname{Re} z| \leqslant \kappa$ *we have* $L(f;z) = O_{\kappa,\varepsilon}(1+|\operatorname{Im} z|^{\varepsilon})$.

Proof. First let us prove that $L(f;s)$ is entire. Let κ be given, and $\mathcal{B}$ a disk of radius κ around 0. By assumption (1.3), $f(p) = o(\log p)$. Therefore there is a $C := C(\kappa) > 2$ such that for all $p \geqslant C$ we have $e^{sf(p)} \leqslant p^{1/3}$ for $\operatorname{Re} s \leqslant \kappa$. In particular none of the terms $(1+e^{sf(p)}/p - 1/p)$ vanish when $\operatorname{Re} s \leqslant \kappa$ and $p > C$. To show that $L(f;s)$ is entire, it's enough to show that the products

$$\prod_{C\leqslant p\leqslant x}\left(1-\frac{1}{p}\right)^{\widehat{\Psi}(f;s)}\cdot\left(1+\frac{e^{sf(p)}}{p-1}\right) \tag{4.3}$$

converge uniformly in $s \in \mathcal{B}$. Equivalently since none of the terms in (4.3) vanish when $\operatorname{Re} s \leqslant \kappa$ (hence in $s \in \mathcal{B}$), it's enough to show that the tails

$$\sum_{p>x}\left[\widehat{\Psi}(f;s)\cdot\log\left(1-\frac{1}{p}\right)+\log\left(1+\frac{e^{sf(p)}}{p-1}\right)\right]\to 0 \tag{4.4}$$

uniformly in $s \in \mathcal{B}$ as $x \to \infty$. Since $|e^{sf(p)}| \leqslant p^{1/3}$ and $|\widehat{\Psi}(f;s)| \leqslant \widehat{\Psi}(f;\kappa)$ for $s \in \mathcal{B}$, it follows from a Taylor expansion that

$$\begin{aligned}&\sum_{p>x}\left[\widehat{\Psi}(f;s)\cdot\log\left(1-\frac{1}{p}\right)+\log\left(1+\frac{e^{sf(p)}}{p-1}\right)\right]\\ =\ &\sum_{p>x}\left[\frac{e^{sf(p)}}{p}-\frac{\widehat{\Psi}(f;s)}{p}\right]+O\left(\sum_{p>x}\frac{\widehat{\Psi}(f;\kappa)}{p^2}+\sum_{p>x}\frac{e^{\kappa f(p)}}{p^2}\right)\end{aligned} \tag{4.5}$$

By lemma 4.3 and an integration by parts the error term in (4.5) is $O_\kappa(1/x)$. We can assume that $x \geqslant e^{e^\kappa}$. Let $F(s;x) = (1/x)\sum_{n\leqslant x} e^{sf(p)}$. We have

$$\begin{aligned}\sum_{p>x}\left[\frac{e^{sf(p)}}{p} - \frac{\widehat{\Psi}(f;s)}{p}\right] &= \int_x^\infty \frac{1}{t}\mathrm{d}\left[F(s;x) - \widehat{\Psi}(f;s)\pi(t)\right] \qquad (4.6)\\ &= -\frac{F(s;x)-\widehat{\Psi}(f;s)\pi(x)}{x} + \int_x^\infty \frac{1}{t^2}\cdot\left[F(s;t) - \widehat{\Psi}(f;s)\pi(t)\right]\mathrm{d}t\end{aligned}$$

Since $t \geqslant x \geqslant e^{e^\kappa}$ by lemma 4.3 we have $F(s;t) - \widehat{\Psi}(f;s)\pi(t) = O_{\kappa,A}\left(t(\log t)^{-A}\right)$ uniformly in $|\operatorname{Im} s| \leqslant \kappa$ and $|\operatorname{Re} s| \leqslant \kappa$ (hence uniformly in $s \in \mathcal{B}$). It follows that (4.6) is bounded by

$$(\log x)^{-A} + \int_x^\infty \frac{t(\log t)^{-A}}{t^2}\mathrm{d}t \ll (\log x)^{-A+1}$$

uniformly in $s \in \mathcal{B}$. Therefore (4.4) holds uniformly in $s \in \mathcal{B}$, and thus

$$\sum_{p\geqslant C}\left[\widehat{\Psi}(f;s)\cdot\log\left(1-\frac{1}{p}\right) + \log\left(1+\frac{e^{sf(p)}}{p-1}\right)\right]$$

is analytic in $\mathcal{B}$. Exponentiating and multiplying by a product over the primes $p \leqslant C \leqslant x$ (obviously analytic) we conclude that

$$L(f;s) \;:=\; \prod_p \left(1-\frac{1}{p}\right)^{\widehat{\Psi}(f;s)}\cdot\left(1+\frac{e^{sf(p)}}{p-1}\right)$$

is analytic in $\mathcal{B}$. Since $\mathcal{B}$ was a ball with an arbitrary radius, it follows that the function $L(f;s)$ is entire. In fact we proved more. We established that the tails in (4.4) are $\ll (\log x)^{-A+1}$. Therefore, for all x large enough (how large x we have to choose depends only on how big $|\operatorname{Re} s|$ we allow)

$$\prod_{p\leqslant x}\left(1-\frac{1}{p}\right)^{\widehat{\Psi}(f;s)}\cdot\left(1+\frac{e^{sf(p)}}{p-1}\right) = L(f;s)\cdot\left(1+O_A\left((\log x)^{-A+1}\right)\right) \qquad (4.7)$$

uniformly in $s \in \mathcal{B}$. In fact in (4.6), $F(s;t) - \widehat{\Psi}(f;s)\pi(t) = O_A(t(\log t)^{-A})$ does hold uniformly in the range $|\operatorname{Re} s| \leqslant \kappa$ and $|\operatorname{Im} s| \leqslant \log\log x$ for $t \geqslant x$, by lemma 4.3. Therefore the tails (4.4) are $O_{\kappa,A}((\log x)^{-A+1})$ uniformly in $|\operatorname{Re} s| \leqslant \kappa$ and $|\operatorname{Im} s| \leqslant \log\log x$ and it follows that (4.7) holds in that range. This gives the second claim of the lemma. Now it remains to prove that $L(f;s) = O_{\kappa,\varepsilon}(1+|\operatorname{Im} s|^\varepsilon)$ uniformly in $|\operatorname{Re} s| \leqslant \kappa$. Let as usual $C := C(\kappa) > 0$ be chosen so that $(1+e^{sf(p)}/(p-1))$ does not vanish in the half-plane $\operatorname{Re} s \leqslant \kappa$ for $p > C$. We want to give a bound for $L(f;s)$ that holds uniformly in $|\operatorname{Re} s| \leqslant \kappa$ and $|\operatorname{Im} s| \leqslant T$.

Without loss of generality $T \geqslant 1$. Uniformly in $|\operatorname{Re} s| \leqslant \kappa$

$$\begin{aligned}
&\sum_{p \geqslant C} \left[\widehat{\Psi}(f;s) \cdot \log \left(1 - \frac{1}{p}\right) + \log\left(1 + \frac{e^{sf(p)}}{p-1}\right)\right] \\
= &\sum_{p \geqslant 3/2} \left[\frac{e^{sf(p)}}{p} - \frac{\widehat{\Psi}(f;s)}{p}\right] + O_\kappa(1) \\
= &\int_{3/2}^{\infty} \frac{F(s;t) - \widehat{\Psi}(f;s)\pi(t)}{t} + O_\kappa(1) \qquad (4.8)
\end{aligned}$$

with $F(s;t) := \sum_{n \leqslant t} e^{sf(p)}$ as usual. Note that by lemma 4.3, for any given $A > 0$ we have uniformly in $|\operatorname{Re} s| \leqslant \kappa$ and $|\operatorname{Im} s| \leqslant T$

$$F(s;t) - \widehat{\Psi}(f;s)\pi(t) = O_{\kappa,A}\left(t(\log t)^{-A}\right) \text{ when } t \geqslant \exp\left(T^{1/A}\right) \qquad (4.9)$$

We split the integral in (4.8) into two parts. The part over $3/2 \leqslant t \leqslant \exp(T^{1/A})$ and the remaining part over $t \geqslant \exp(T^{1/A})$. Note that $|F(s;t)| \leqslant F(\kappa;t)$. Furthermore by lemma 4.3, $F(\kappa;t) \ll \widehat{\Psi}(f;\kappa)\pi(t)$. Using these observations the integral over the $3/2 \leqslant t \leqslant \exp(T^{1/A})$ part is bounded by

$$\int_{3/2}^{e^{T^{1/A}}} \frac{1}{t^2}\left[F(\kappa;t) + \widehat{\Psi}(f;\kappa)\pi(t)\right] dt \ll \widehat{\Psi}(f;\kappa) \sum_{p \leqslant e^{T^{1/A}}} \frac{1}{p} \ll \frac{\widehat{\Psi}(f;\kappa)}{A}\log(1+T) \qquad (4.10)$$

by making A large enough we can make the integral above $\leqslant \varepsilon \log(1+T)$ for any given $\varepsilon > 0$. The remaining integral over $t \geqslant \exp(T^{1/A})$ is bounded using (4.9). Indeed we find that

$$\int_{e^{T^{1/A}}}^{\infty} \frac{1}{t^2} \cdot \left[F(s;t) - \widehat{\Psi}(f;s)\pi(t)\right] dt \ll_{\kappa,A} \int_{e^{T^{1/A}}}^{\infty} \frac{t \cdot (\log t)^{-A}}{t^2} dt \ll_{\kappa,A} T^{-1+1/A} \qquad (4.11)$$

Of course we can assume that $A \geqslant 2$. By (4.10) and (4.11) we conclude that the integral in (4.8) is $\leqslant \varepsilon\log(1+T) + O_\kappa(1)$ uniformly in $|\operatorname{Re} s| \leqslant \kappa$ and $|\operatorname{Im} s| \leqslant T$. Exponentiating (4.8) it follows that uniformly in $|\operatorname{Re} s| \leqslant \kappa$ and $|\operatorname{Im} s| \leqslant T$,

$$\prod_{C \leqslant p} \left(1 - \frac{1}{p}\right)^{\widehat{\Psi}(f;s)} \cdot \left(1 + \frac{e^{sf(p)}}{p-1}\right) = O_{\kappa,\varepsilon}(1 + T^{\varepsilon})$$

Multiplying on both sides by $\prod_{p<C}(1-1/p)^{\widehat{\Psi}(f;s)} \cdot (1 + e^{sf(p)}/(p-1))$ does not change the bound. Thus $L(f;s) = O_{\kappa,\varepsilon}(1+T^{\varepsilon})$ uniformly in $|\operatorname{Re} s| \leqslant \kappa, |\operatorname{Im} s| \leqslant T$ in particular $L(f;s) = O_{\kappa,\varepsilon}(1 + |\operatorname{Im} s|^{\varepsilon})$ uniformly in $|\operatorname{Re} s| \leqslant \kappa$. The claim follows. □

Finally we need an elementary lemma on sums of multiplicative functions. The following lemma appears on page 308 of Tenenbaum's book [17].

Lemma 4.5. *Let $g \geqslant 0$ be a multiplicative function, such that for some A and B,*

$$\sum_{p \leqslant x} g(p) \log p \leqslant Ax$$

$$\sum_{p} \sum_{\nu \geqslant 2} \frac{g(p^{\nu})}{p^{\nu}} \cdot \log p^{\nu} \leqslant B$$

Then, for $x > 1$,

$$\sum_{n \leqslant x} g(n) \leqslant (A + B + 1) \cdot \frac{x}{\log x} \sum_{n \leqslant x} \frac{g(n)}{n}$$

Corollary 4.6. *Let $f \in \mathcal{C}$. Given $C > 0$, uniformly in $0 \leqslant \kappa \leqslant C$,*

$$\sum_{n \leqslant x} e^{\kappa f(n)} = O_C\left(x \cdot (\log x)^{\widehat{\Psi}(f;\kappa) - 1}\right)$$

$$\sum_{n \leqslant x} \frac{e^{\kappa f(n)}}{n} = O_C\left((\log x)^{\widehat{\Psi}(f;\kappa)}\right)$$

Proof. In lemma 4.5 we choose $g(n) := e^{\kappa f(n)}$. By lemma 4.3 there is an $A := A(C)$ such that

$$\sum_{p \leqslant x} e^{\kappa f(p)} \cdot \log p \leqslant \sum_{p \leqslant x} e^{C f(p)} \cdot \log p \leqslant A(C) \cdot x$$

for all $x > 1$. Also note that

$$\sum_{p} \sum_{\nu \geqslant 2} \frac{e^{\kappa f(p^{\nu})}}{p^{\nu}} \cdot \log p^{\nu} \ll \sum_{p} e^{C f(p)} \cdot \frac{\log p}{p^2}$$

and by lemma 4.3 the above sum converges. Hence the second assumption of the lemma holds, for some $B := B(C)$ large enough. Thus by lemma 4.5,

$$\sum_{n \leqslant x} e^{\kappa f(n)} = O_C\left(\frac{x}{\log x} \cdot \sum_{n \leqslant x} \frac{e^{\kappa f(n)}}{n}\right) = O\left(\frac{x}{\log x} \cdot \prod_{p \leqslant x}\left(1 + \frac{e^{\kappa f(p)}}{p - 1}\right)\right) \tag{4.12}$$

By lemma 4.3 and an integration by parts $\sum_{p \leqslant x} e^{\kappa f(p)} \cdot (p-1)^{-1} = \widehat{\Psi}(f;\kappa) \log\log x + O_C(1)$. Therefore the product in (4.12) is bounded by $(\log x)^{\widehat{\Psi}(f;\kappa)}$. Hence the mean-value $M(x) := \sum_{n \leqslant x} e^{\kappa f(n)} \ll x(\log x)^{\widehat{\Psi}(f;\kappa) - 1}$ and also

$$\begin{aligned}\sum_{n \leqslant x} \frac{e^{\kappa f(n)}}{n} &= \frac{M(x)}{x} + \int_1^x \frac{M(t)}{t^2} \cdot dt \\ &\ll (\log x)^{\widehat{\Psi}(f;\kappa) - 1} + \int_1^x (\log t)^{\widehat{\Psi}(f;\kappa) - 1} \cdot t^{-1} dt \ll (\log x)^{\widehat{\Psi}(f;\kappa)}\end{aligned}$$

as desired. ∎

We are now ready to prove Proposition 4.1.

Proof of Proposition 4.1. Let P_k denote the product of the first k primes. The plan of our proof is the following. First we estimate

$$m_k(x; z) := \sum_{\substack{n \leqslant x \\ (n, P_k) = 1}} \frac{e^{zf(n)}}{n} \tag{4.13}$$

uniformly in $|\operatorname{Re} z| \leqslant C$ and with $k = k(C) > 0$ chosen suitably. Then we relate (4.13) to the mean value $M_k(x; z) := \sum_{n \leqslant x, (n, P_k) = 1} e^{zf(n)}$. By a simple convolution argument we subsequently obtain the desired asymptotic for $M(x; z) := \sum_{n \leqslant x} e^{zf(n)}$. Denote by $\Lambda_f(z; n)$ the "generalized van Mangoldt function" defined by

$$e^{zf(n)} \cdot \log n = \sum_{d | n} e^{zf(d)} \cdot \Lambda_f(z; n/d) \tag{4.14}$$

Looking at the Dirichlet series for $\Lambda_f(z; n)$ we conclude that $\Lambda_f(z; n)$ vanishes when n is not a prime power. On the other hand when $n = p^\alpha$ is a prime power (see [7], lemma 1.1.2)

$$\Lambda_f(z; p^\alpha) = \log p^\alpha \cdot \sum_{m \leqslant \alpha} \frac{(-1)^{m-1}}{m} \cdot e^{zmf(p)} \cdot \binom{\alpha - 1}{m - 1}$$

Therefore

$$\begin{aligned} \sum_{\substack{n \leqslant x \\ (n, P_k) = 1}} \Lambda_f(z; n) &= \sum_{\substack{p^\alpha \leqslant x \\ p > k}} \log p^\alpha \sum_{m \leqslant \alpha} \frac{(-1)^{m-1}}{m} \cdot e^{zmf(p)} \cdot \binom{\alpha - 1}{m - 1} \\ &= \sum_{m \leqslant \log x / \log k} \frac{(-1)^{m-1}}{m} \cdot \sum_{\substack{p^\alpha \leqslant x \\ \alpha \geqslant m \\ p > k}} \log p^\alpha \cdot e^{zmf(p)} \cdot \binom{\alpha - 1}{m - 1} \end{aligned} \tag{4.15}$$

We split the above sum into two. The terms with $m = 1$ contribute

$$\sum_{\substack{p^\alpha \leqslant x \\ p > k}} e^{zf(p)} \cdot \log p^\alpha = \widehat{\Psi}(f; z) \cdot x + O_{A,C}\left(x \cdot \frac{1 + |\operatorname{Im} z|}{(\log x)^{3A}} \right) \tag{4.16}$$

by lemma 4.3 and an integration by parts (using the prime number theorem with a $O_B(x(\log x)^{-B})$ error term. The terms $m \geqslant 2$ contribute

$$\sum_{2\leqslant m\leqslant \log x/\log k} \frac{(-1)^{m-1}}{m} \sum_{\substack{p^\alpha \leqslant x \\ \alpha \geqslant m \\ p > k}} \alpha \log p \cdot e^{zmf(p)} \cdot \binom{\alpha - 1}{m-1} \tag{4.17}$$

Since $f(p) = o(\log p)$ (because of assumption (1.3)) we can choose k large enough so as to have $f(p) \leqslant (1/4C)\log p$ for all $p > k$. With this choice of $k := k(C)$, for $\operatorname{Re} z \leqslant C$, the sum in (4.17) is bounded in modulus by

$$\begin{aligned} &\ll \sum_{2\leqslant m\leqslant \log x} \frac{1}{m} \sum_{\substack{p^\alpha \leqslant x \\ \alpha \geqslant m \\ p > k}} (\log x)^2 \cdot \exp\left(Cm \cdot \frac{\log p}{4C}\right) \cdot 2^\alpha \\ &\ll x^{\log 2/\log k} \cdot \sum_{2\leqslant m\leqslant \log x} \frac{1}{m} \cdot (\log x)^3 \cdot x^{1/4} \cdot x^{1/m} \ll x^{3/4+\log 2/\log k} \cdot (\log x)^4 \end{aligned}$$

To obtain the second bound we use $p \leqslant x^{1/m}$ to get $\exp(Cm\log p/4C) \leqslant x^{1/4}$ and then the bound $\sum_{p^\alpha \leqslant x, \alpha\geqslant m} 1 \ll x^{1/m}\log x$. Making k larger if necessary we see that the sum in (4.17) is bounded by $x^{1-\varepsilon}$ for some small but fixed $\varepsilon > 0$. Our bound for (4.17) together with (4.16) allows us to conclude that

$$\sum_{\substack{n \leqslant x \\ (n, P_k) = 1}} \Lambda_f(z; n) = \hat{\Psi}(f; z) \cdot x + O_{A,C}\left(x \cdot \frac{1 + |\operatorname{Im} z|}{(\log 2x)^{3A}}\right) \tag{4.18}$$

Upon integrating by parts (and making A larger if necessary) we obtain

$$\sum_{\substack{n \leqslant x \\ (n, P_k) = 1}} \frac{\Lambda_f(z; n)}{n} = \hat{\Psi}(f; z) \cdot \log x + A_0(f; z) + O_{A,C}\left(\frac{1 + |\operatorname{Im} z|}{(\log 2x)^{3A}}\right) \tag{4.19}$$

uniformly in $|\operatorname{Re} z| \leqslant C$ where $A_0(f; z) := \int_1^\infty [G(z; t) - \hat{\Psi}(f; z)t]t^{-2}dt$ is analytic in $|\operatorname{Re} z| \leqslant C$ and where $G(z; t) := \sum_{n\leqslant x, (n, P_k)} \Lambda_f(z; n)$. Using equation (4.16) and repeating the same proof as in lemma 4.4 we find that $A_0(f; z) = O_{A,C}(1 + |\operatorname{Im} z|^{1/A})$ uniformly in the range $|\operatorname{Re} z| \leqslant C$. Following Levin and Fainleb we express $\sum_{n\leqslant x, (n, P_k) = 1} e^{zf(n)} \cdot \log n \cdot n^{-1}$ in two different ways. On the one hand, integrating by parts we get

$$\sum_{\substack{n \leqslant x \\ (n, P_k) = 1}} \frac{e^{zf(n)} \cdot \log n}{n} = m_k(x; z) \cdot \log x - \int_2^x \frac{m_k(u; z)}{u} du \tag{4.20}$$

where $m_k(x;z) := \sum_{n\leqslant x,(n,P_k)=1} e^{zf(n)}\cdot n^{-1}$. On the other by (4.14) and (4.19),

$$\begin{aligned}\sum_{\substack{n\leqslant x\\(n,P_k)=1}}\frac{e^{zf(n)}\cdot\log n}{n} &= \sum_{\substack{d\leqslant x\\(d,P_k)=1}}\frac{e^{zf(d)}}{d}\sum_{\substack{n\leqslant x/d\\(n,P_k)=1}}\frac{\Lambda_f(z;n)}{n}\\ &= \sum_{\substack{d\leqslant x\\(d,P_k)=1}}\frac{e^{zf(d)}}{d}\cdot\left[\widehat{\Psi}(f;z)\cdot(\log x-\log d)+A_0(f;z)+O_{A,C}\left(\frac{1+|\operatorname{Im} z|}{(\log 2x/d)^{3A}}\right)\right]\\ &= \widehat{\Psi}(f;z)\int_2^x\frac{m_k(u;z)}{u}du+A_0(f;z)m_k(x;z)+O_{A,C}\left(\frac{1+|\operatorname{Im} z|}{(\log x)^{2A}}\right)\end{aligned}\tag{4.21}$$

In the error term we bound $\sum e^{\kappa f(d)}\cdot d^{-1/2}\cdot d^{-1/2}\cdot(\log 2x/d)^{-3A}$ by using Cauchy-Schwarz's inequality and Corollary 4.6 (also, we assume without loss of generality that A is chosen sufficiently large, $A\geqslant 4\widehat{\Psi}(f;2C)+4$ will do). Comparing (4.20) with (4.21) we conclude that

$$m_k(x;z)\log x-(1+\widehat{\Psi}(f;z))\int_2^x\frac{m_k(u;z)}{u}du=A_0(f;z)m_k(x;z)+O\left(\frac{1+|\operatorname{Im} z|}{(\log x)^{2A}}\right)$$

uniformly in $|\operatorname{Re} z|\leqslant C$. Recall that A is taken large enough, $A\geqslant 4\widehat{\Psi}(f;2C)+4$. Dividing by $x(\log x)^{\widehat{\Psi}(f;z)+2}$ on both sides and integrating from 2 to x we obtain

$$\begin{aligned}&\int_2^x\frac{m_k(u;z)du}{u(\log u)^{\widehat{\Psi}(f;z)+1}}-\int_2^x\frac{1+\widehat{\Psi}(f;z)}{u(\log u)^{\widehat{\Psi}(f;z)+2}}\int_2^u\frac{m_k(v;z)}{v}dvdu\\ &= A_0(f;z)\int_2^x\frac{m_k(x;u)du}{u(\log u)^{\widehat{\Psi}(f;z)+2}}+A_1(f;z)+O\left(\frac{1+|\operatorname{Im} z|}{(\log x)^{A+\widehat{\Psi}(f;C)+1}}\right)\end{aligned}\tag{4.22}$$

with both $A_0(f;z)$ and $A_1(f;z)$ analytic in $|\operatorname{Re} z|\leqslant C$. In fact by a proof similar to the one in lemma 4.4 we find that $A_1(f;z)\ll_{A,C} 1+|\operatorname{Im} z|^{1/A}$. Upon interchanging integrals the second term in (4.22) can be re-written as

$$\begin{aligned}&\big(1+\widehat{\Psi}(f;z)\big)\int_2^x\frac{m_k(v;z)}{v}\int_v^x\frac{dudv}{u(\log u)^{\widehat{\Psi}(f;z)+2}}\\ &= \int_2^x\frac{m_k(v;z)dv}{v(\log v)^{\widehat{\Psi}(f;z)+1}}-\int_2^x\frac{m_k(v;z)dv}{v(\log x)^{\widehat{\Psi}(f;z)+1}}\end{aligned}$$

Therefore (4.22) simplifies to

$$\begin{aligned}\int_2^x\frac{m_k(u;z)}{u}du &= A_0(f;z)\int_2^x\frac{m_k(u;z)}{u(\log u)^{\widehat{\Psi}(f;z)+2}}\cdot(\log x)^{\widehat{\Psi}(f;z)+1}\\ &\quad +A_1(f;z)\cdot(\log x)^{\widehat{\Psi}(f;z)+1}+O_{A,C}\left(\frac{1+|\operatorname{Im} z|}{(\log x)^{A}}\right)\end{aligned}$$

Plugging the above relation into the equation right above (4.22) yields

$$\begin{aligned}
m_k(x;z)\cdot\log x &= (1+\widehat{\Psi}(f;z))A_0(f;z)\int_2^x \frac{m_k(u;z)du}{u(\log u)^{\widehat{\Psi}(f;z)+2}}\cdot(\log x)^{\widehat{\Psi}(f;z)+1} \\
&\quad +(1+\widehat{\Psi}(f;z))A_1(f;z)\cdot(\log x)^{\widehat{\Psi}(f;z)+1}+O\left(\frac{1+|\operatorname{Im} z|}{(\log x)^A}\right) \\
&\quad +A_0(f;z)\cdot m_k(x;z)+O\left(\frac{1+|\operatorname{Im} z|}{(\log x)^{2A}}\right)
\end{aligned}$$

because $|\widehat{\Psi}(f;z)|\leqslant\widehat{\Psi}(f;C)$. We could iterate to obtain an asymptotic expansion. We choose not to do so. Instead we note the bound $|m_k(x;z)|\leqslant m_k(x;\kappa)\ll(\log x)^{\widehat{\Psi}(f;\kappa)}$ ($\kappa:=\operatorname{Re} z$) coming from from Corollary 4.6. Recall also that $A_0(f;z)\ll_C 1+|\operatorname{Im} z|^{1/A}$ and that $\widehat{\Psi}(f;z)\ll_C 1$. With these two bounds at hand our previous equality becomes

$$m_k(x;z) = (1+\widehat{\Psi}(f;z))A_1(f;z)\cdot(\log x)^{\widehat{\Psi}(f;z)}+O\left(\mathcal{E}_A(x;z)\cdot(\log x)^{\widehat{\Psi}(f;\kappa)-1}\right)$$

uniformly in $|\operatorname{Re} z|\leqslant C$ and where $\mathcal{E}_A(z;x)=1+|\operatorname{Im} z|^{1/A}+|\operatorname{Im} z|\cdot(\log x)^{-1}$. We now evaluate $M_k(x;z):=\sum_{n\leqslant x,(n,P_k)=1}e^{zf(n)}$. Using the definition of $\Lambda_f(z;n)$, equation (4.18), corollary 4.6, and the previous line, we get

$$\begin{aligned}
W_k(x;z) &= \sum_{\substack{n\leqslant x\\(n,P_k)=1}} e^{zf(n)}\cdot\log n = \sum_{\substack{d\leqslant x\\(d,P_k)=1}} e^{zf(d)}\sum_{\substack{n\leqslant x/d\\(n,P_k)=1}}\Lambda_f(z;n) \\
&= \sum_{\substack{d\leqslant x\\(d,P_k)=1}} e^{zf(d)}\cdot\left[\widehat{\Psi}(f;z)(x/d)+O_{A,C}\left(\frac{x}{d}\cdot\frac{1+|\operatorname{Im} z|}{(\log 2x/d)^{2A}}\right)\right] \\
&= \widehat{\Psi}(f;z)\cdot x m_k(x;z)+O_{A,C}\left(x\cdot\frac{1+|\operatorname{Im} z|}{(\log x)^A}\right) \\
&= A_2(f;z)\cdot x(\log x)^{\widehat{\Psi}(f;z)}+O\left(\mathcal{E}_A(x;z)\cdot x(\log x)^{\widehat{\Psi}(f;\kappa)-1}\right)
\end{aligned}$$

uniformly in $|\operatorname{Re} z|\leqslant C$ and where $A_2(f;z):=(1+\widehat{\Psi}(f;z))\widehat{\Psi}(f;z)A_1(f;z)$. In the second line above, we bound $\sum e^{\kappa f(d)}\cdot d^{-1}\cdot(\log 2x/d)^{-2A}$ by applying Cauchy-Schwarz's inequality and using Corollary 4.6 (also recall that $A\geqslant 4\widehat{\Psi}(f;2C)+4$). Integrating by parts our previous result we conclude that the mean value $M_k(x;z)$ equals to

$$M_k(x;z):=\int_2^x\frac{dW_k(t;z)}{\log t}=\frac{W_k(x;z)}{\log x}+\int_2^x\frac{W_k(t;z)}{t(\log t)^2}dt$$

Corollary 4.6 yields the bound $|W_k(t;z)|\leqslant W_k(t;\kappa)=O_C(t\cdot(\log t)^{\widehat{\Psi}(f;\kappa)})$ where as usual $\kappa:=\operatorname{Re} z$. It follows that the second integral in the above equation is bounded by $x\,\cdot$

$(\log x)^{\widehat{\Psi}(f;\kappa)-2}$. We conclude that

$$M_k(x;z) = A_2(f;z)\cdot x(\log x)^{\widehat{\Psi}(f;z)-1} + O_{A,C}\left(\mathcal{E}_A(x;z)\cdot x(\log x)^{\widehat{\Psi}(f;\kappa)-2}\right)$$

It remains to estimate $M(x;z) = \sum_{n\leqslant x} e^{zf(n)}$. At this point recall that the function $A_1(f;z) = O_C(1+|\operatorname{Im} z|^{1/A})$ and that $A_2(f;z) = \widehat{\Psi}(f;z)(1+\widehat{\Psi}(f;z))A_1(f;z)$ hence the same bound holds for $A_2(f;z)$. Let $g(z;n)$ be a multiplicative function defined by $g(z;p^\ell) = \exp(zf(p^\ell))$ when $p\leqslant k$ and $g(z;p^\ell) = 0$ otherwise. We have

$$M(x;z) = \sum_{d\leqslant x} g(z;d) \sum_{\substack{n\leqslant x/d\\ (n,P_k)=1}} e^{zf(n)} = \sum_{d\leqslant x} g(z;d) M_k(x/d;z)$$

Using our estimate for $M_k(x/d;z)$ this simplifies to

$$M(x;z) = A_3(f;z)\cdot x(\log x)^{\widehat{\Psi}(f;z)-1} + O_{A,C}\left(\mathcal{E}_A(x;z)\cdot x(\log x)^{\widehat{\Psi}(f;\kappa)-2}\right)$$

where $A_3(f;z) = \prod_{p\leqslant k}(1+e^{zf(p)}\cdot(p-1)^{-1})A_2(f;z)$ is analytic. It remains to show that $A_3(f;z) = L(f;z)/\Gamma(\widehat{\Psi}(f;z))$. Here, we use an abelian argument. Consider the two-variable function.

$$L_f(s;z) := \prod_p \left(1-\frac{1}{p^s}\right)^{\widehat{\Psi}(f;z)}\cdot\left(1+\frac{e^{zf(p)}}{p^s-1}\right)$$

Mimicking the proof of lemma 4.4 it is not too hard to prove that $L_f(s;\kappa)$ is uniformly bounded when $1\leqslant s\leqslant 2$ and $\kappa\in[0;\delta]$ for some $\delta>0$. In addition by [7] (corollary to lemma 1.1.7) for fixed $\kappa\geqslant 0$ the function $L_f(s;\kappa)$ is right continuous at $s=1$, when s is going through the reals. Thus $L_f(s;\kappa)\to L_f(1;\kappa) = L(f;\kappa)$ for fixed κ and as $s\to 1^+$. Furthermore we have the factorization

$$L_f(s;\kappa)\zeta(s)^{\widehat{\Psi}(f;\kappa)} = \sum_{n\geqslant 1}\frac{e^{\kappa f(n)}}{n^s} = s\int_1^\infty M(t;\kappa)t^{-s-1}dt \tag{4.23}$$

$$= \quad A_3(f;\kappa)\cdot s\int_1^\infty (\log t)^{\widehat{\Psi}(f;\kappa)-1}\cdot t^{-s}dt + O_\delta\left(\int_1^\infty (\log t)^{\widehat{\Psi}(f;\kappa)-2}\cdot t^{-s}dt\right)$$

By a change of variable $u := \log t$ the first integral becomes

$$\int_1^\infty (\log t)^{\widehat{\Psi}(f;\kappa)-1}\cdot t^{-s}dt = \int_0^\infty e^{-t(s-1)}\cdot t^{\widehat{\Psi}(f;\kappa)-1}dt = \frac{\Gamma(\widehat{\Psi}(f;\kappa))}{(s-1)^{\widehat{\Psi}(f;\kappa)}}$$

Therefore (4.23) can be re-written as

$$L_f(s;\kappa)\cdot\zeta(s)^{\widehat{\Psi}(f;\kappa)} = A_3(f;\kappa)\Gamma(\widehat{\Psi}(f;\kappa))s(s-1)^{-\widehat{\Psi}(f;\kappa)} + O((s-1)^{-\widehat{\Psi}(f;\kappa)+1})$$

Choose $s = 1+1/\log x$ and fix κ. By our earlier remark $L_f(s;\kappa) = L(f;\kappa)+o(1)$. Furthermore $\zeta(s)\sim 1/(s-1)$. Therefore the previous equation turns into

$$L(f;\kappa) - A_3(f;\kappa)\Gamma(\widehat{\Psi}(f;\kappa)) = o(1)$$

It follows that $A_3(f;\kappa) = L(f;\kappa)/\Gamma(\widehat{\Psi}(f;\kappa))$. Since both functions are analytic in $|\operatorname{Re} z| \leqslant C$ and coincide on a compact interval we get $A_3(f;z) = L(f;z)/\Gamma(\widehat{\Psi}(f;z))$ for all $|\operatorname{Re} z| \leqslant C$. It now follows that

$$\frac{1}{x}\sum_{n\leqslant x} e^{zf(n)} = \frac{L(f;z)}{\Gamma(\widehat{\Psi}(f;z))}\cdot(\log x)^{\widehat{\Psi}(f;z)-1} + O_{A,C}\left(\mathcal{E}_A(x;z)\cdot(\log x)^{\widehat{\Psi}(f;\kappa)-2}\right)$$

uniformly in $|\operatorname{Re} z| \leqslant C$ which is the desired claim. $\square$

4.2. **Two simple estimates for $\nu_f(x;\Delta)$.** In the next lemma we collect a few useful facts about $\nu_f(x;\Delta)$. First we prove that $\nu_f(x;\Delta)$ is essentially $\Delta/\sigma_\Psi(f;x)$.

Lemma 4.7. *Let $f \in \mathcal{C}$. Given $\delta > 0$ uniformly in $1 \leqslant \Delta \leqslant \delta\sigma_\Psi(f;x)$,*

$$\nu_f(x;\Delta) \asymp_\delta \Delta/\sigma_\Psi(f;x)$$

Furthermore $\nu_f(x;\Delta) \sim \Delta/\sigma_\Psi(f;x)$ in the $1 \leqslant \Delta \leqslant o(\sigma_\Psi(f;x))$ range. Finally the function $\omega(f;z)$ is analytic in a neighborhood of $\mathbb{R}^+ \cup \{0\}$

Proof. Consider the function $\omega(f;z)$ defined implicitly by

$$\widehat{\Psi}'(f;\omega(f;z)) = \widehat{\Psi}'(f;0) + z\cdot\widehat{\Psi}''(f;0)$$

Note that by definition $\nu = \nu_f(x;\Delta) := \omega(f;\Delta/\sigma_\Psi(f;x))$. Since $\widehat{\Psi}''(f;x) \neq 0$ for all $x \geqslant 0$, by Lagrange's inversion the function $\omega(f;z)$ is analytic in a neighborhood of $\mathbb{R}^+ \cup \{0\}$. Therefore

$$\nu_f(x;\Delta) \;=\; \omega(f;\Delta/\sigma_\Psi(f;x)) = \Delta/\sigma_\Psi(f;x) + O\left((\Delta/\sigma_\Psi(f;x)^2)\right) \tag{4.24}$$

Therefore for $\Delta \leqslant c\sigma_\Psi(f;x)$ and c small enough $\nu_f(x;\Delta) \asymp \Delta/\sigma_\Psi(f;x)$. In the remaining range $c \leqslant \Delta/\sigma_\Psi(f;x) \leqslant \delta$ it is clear that $\nu_f(x;\Delta) \asymp 1 \asymp \Delta/\sigma_\Psi(f;x)$: indeed, $\nu_f(x;\Delta) = \omega(f;\Delta/\sigma_\Psi(f;x))$, the function $\omega(f;x)$ is positive and continuous for $x \geqslant 0$ and $\Delta/\sigma_\Psi(f;x)$ belongs to a bounded interval. Also, the second assertion of the lemma follows immediately from (4.24). $\square$

Lemma 4.8. *Let $f \in \mathcal{C}$. As usual let $\xi_f(x;\Delta) := \mu(f;x) + \Delta\sigma(f;x)$. For any given $\delta > 0$, we have uniformly in $1 \leqslant \Delta \leqslant \delta\sigma(f;x)$,*

$$\xi_f(x;\Delta) \;=\; \widehat{\Psi}'(f;\nu_f(x;\Delta))\cdot\log\log x + c(f) + O_\delta\left(\frac{1}{\sqrt{\log\log x}}\right)$$

Proof. Integrating by parts the result of lemma 4.3 gives an estimate for the average $\sum_{p\leqslant x} e^{sf(p)}/p$. Differentiating using Cauchy's formula and setting $s = 0$ we find that

$$\begin{aligned}\mu(f;x) \;&=\; \widehat{\Psi}'(f;0)\cdot\log\log x + c(f) + O\left(\frac{1}{\sqrt{\log x}}\right)\\ \sigma^2(f;x) \;&=\; \widehat{\Psi}''(f;0)\cdot\log\log x + O(1)\end{aligned}$$

By definition of $\nu_f(x;\Delta)$ we have

$$\begin{aligned}\widehat{\Psi}'(f;\nu_f(x;\Delta))\cdot\operatorname{loglog}x &= \widehat{\Psi}'(f;0)\cdot\operatorname{loglog}x+\Delta(\widehat{\Psi}''(f;0)\operatorname{loglog}x)^{1/2}\\ &= \mu(f;x)-c(f)+\Delta\sigma(f;x)+O\left(\frac{\Delta}{\operatorname{loglog}x}\right)\\ &= \mu(f;x)-c(f)+\Delta\sigma(f;x)+O_\delta\left(\frac{1}{\sqrt{\operatorname{loglog}x}}\right)\end{aligned}$$

and the claim follows. $\square$

4.3. **Large deviations when** $1\leqslant\Delta=o((\operatorname{loglog}x)^{1/6})$**.** The following is a consequence of a result of Hwang [11] (see the statement of the main result in 1.1 and then Corollary 3).

Proposition 4.9. *Let* $f\in\mathcal{C}$*. Let* $\Omega(f;x)$ *be a sequence of random variables, such that*

$$\mathbb{E}\left[e^{s\Omega(f;x)}\right]=\mathcal{A}(s)\cdot(\log x)^{\widehat{\Psi}(f;s)-1}\cdot(1+o_{x\to\infty}(1))$$

uniformly in $|s|\leqslant\varepsilon$ *for some* $\varepsilon>0$ *sufficiently small and with* $\mathcal{A}(s)$ *analytic and non-zero in a neighborhood of* $s=0$*. Then, uniformly in* $1\leqslant\Delta\leqslant o\left(\sigma(f;x)^{1/3}\right)$*,*

$$\mathbb{P}\left(\frac{\Omega(f;x)-\mu(f;x)}{\sigma(f;x)}\geqslant\Delta\right)\sim\int_\Delta^\infty e^{-u^2/2}\cdot\frac{du}{\sqrt{2\pi}}$$

For all interesting $\Omega(f;x)$ we will be able to determine asymptotics for

$$\mathbb{P}\left(\frac{\Omega(f;x)-\mu(f;x)}{\sigma(f;x)}\geqslant\Delta\right)$$

when Δ is in the range $(\operatorname{loglog}x)^\varepsilon\ll\Delta\leqslant c\sigma(f;x)$. Hwang's lemma will be used to complement these results – that is, handle the (easy) range $1\leqslant\Delta\leqslant o((\operatorname{loglog}x)^{1/6})$. Let us note that Maciulis [14] proved a result similar to proposition 4.9, but much earlier. The drawback of his result is that it is harder to use because of the many parameters introduced in the statement.

4.4. **Large deviations:** $(\operatorname{loglog}x)^\varepsilon\ll\Delta\ll\sigma(f;x)$ **and** $\Psi(f;t)$ **non-lattice.** The object of this section is to prove the following (general) lemma.

Proposition 4.10. *Let* $f\in\mathcal{C}$*. Suppose that* $\Psi(f;t)$ *is not lattice distributed. Let* $\Omega(f;x)$ *be a sequence of random variables such that, for any given* $C>0$*, uniformly in* $0\leqslant\kappa:=\operatorname{Re}s\leqslant C$ *and* $|\operatorname{Im}s|\leqslant\operatorname{loglog}x$*,*

$$\mathbb{E}\left[e^{s\Omega(f;x)}\right]=\mathcal{A}(s)\cdot(\log x)^{\widehat{\Psi}(f;s)-1}+O_C\left((\log x)^{\widehat{\Psi}(f;\kappa)-3/2}\right)$$

Here $\mathcal{A}(s)$ *is analytic in* $\operatorname{Re}s\geqslant0$ *and non-vanishing on* $\mathbb{R}^+\cup\{0\}$*. Assume that* $\mathcal{A}(s)\ll_C(1+|\operatorname{Im}s|^{1/8})$ *holds throughout* $0\leqslant\operatorname{Re}s\leqslant C$*. Then, given* $\delta,\varepsilon>0$*, uniformly in* $(\operatorname{loglog}x)^\varepsilon\ll\Delta\leqslant\delta\sigma(f;x)$*,*

$$\mathbb{P}(\Omega(f;x)\geqslant\mu(f;x)+\Delta\sigma(f;x))\sim\mathcal{A}(\nu)\cdot\frac{(\log x)^{\widehat{\Psi}(f;\nu)-1-\nu\widehat{\Psi}'(f;\nu)}}{\nu(2\pi\widehat{\Psi}''(f;\nu)\operatorname{loglog}x)^{1/2}}\cdot e^{-\nu c(f)}$$

where $\nu := \nu_f(x;\Delta)$ *is the unique positive solution to the equation*

$$\widehat{\Psi}'(f;\nu)\cdot \operatorname{loglog} x = \widehat{\Psi}'(f;0)\cdot \operatorname{loglog} x + \Delta(\widehat{\Psi}''(f;0)\operatorname{loglog} x)^{1/2}$$

and $c(f)$ *as in the statement of Theorem 2.8 (or see section 3).*

It is possible to prove proposition 4.10 using the method of "associated distribution" due to Cramer [2]. The method presented here is more concise, and avoids some of the redundancy inherent in Cramer's method. One of the peculiarity of our method is that it seems to require an asymptotic for $\mathbb{E}[e^{s\Omega(f;x)}]$ in the range $|\operatorname{Re} s| \leqslant C$ and $|\operatorname{Im} s| \leqslant \psi(x)$ for some $\psi(x) \to \infty$, whereas Cramer's methods needs only an assumption on the range $|s| \leqslant C$, for C big enough.

Our proof relies on the following six lemmata. The first lemma is "well-known". A proof can be found in Petrov's book [15] (or in Esséen's thesis [9], theorem 5, p. 26).

Lemma 4.11. *A distribution function* $F(t)$ *is not lattice distributed if and only if for all* $t \neq 0$ *the Fourier transform* $\phi(t) = \int_{\mathbb{R}} e^{itu} dF(u)$ *has modulus* < 1.

Lemma 4.11 admits the following consequence.

Lemma 4.12. *Let* $f \in \mathcal{C}$. *Suppose that* $\Psi(f;t)$ *is not lattice distributed. Then*

$$\phi(t) = e^{\widehat{\Psi}(f;it)-1}$$

is the Fourier transform of a non-lattice distribution function. Furthermore, for any $w \geqslant 0$ *and* $t \in \mathbb{R}$, *we have*

$$|\exp(\widehat{\Psi}(f;w+it) - \widehat{\Psi}(f;w))| \leqslant |\phi(t)|$$

Proof. For a distribution function F denote by F^{*n} the n-fold convolution of F with itself. Consider the distribution function

$$D(f;t) = \frac{1}{e}\sum_{k\geqslant 0} \Psi^{*k}(f;t)\cdot \frac{1}{k!}$$

The Fourier transform of $D(f;t)$ is given by

$$\begin{aligned}\int_{\mathbb{R}} e^{itu} dD(f;u) &= \frac{1}{e}\sum_{k\geqslant 0}\frac{1}{k!}\int_{\mathbb{R}} e^{itu} d\Psi^{*k}(f;u) \\ &= \frac{1}{e}\sum_{k\geqslant 0}\frac{1}{k!}\cdot \widehat{\Psi}(f;it)^k = e^{\widehat{\Psi}(f;it)-1}\end{aligned}$$

This proves existence. Furthermore, since $\Psi(f;t)$ is not lattice distributed, by Lemma 4.11, we have $|\widehat{\Psi}(f;it)| < 1$ for all $t \neq 0$. Therefore $|e^{\widehat{\Psi}(f;it)-1}| < 1$ for all $t \neq 0$. Hence by Lemma 4.11, $e^{\widehat{\Psi}(f;it)-1}$ is the Fourier transform of a non-lattice distribution function. Finally, for the last statement of this lemma, let us note that

$$\begin{aligned}\mathrm{Re}\left(\widehat{\Psi}(f;w+it)-\widehat{\Psi}(f;w)\right) &= \int_0^\infty e^{wu}\cdot(\cos(tu)-1)\,d\Psi(f;u)\\ &\leqslant \int_0^\infty (\cos(tu)-1)d\Psi(f;u) = \mathrm{Re}\left(\widehat{\Psi}(f;it)-1\right)\end{aligned}$$

Note that $\Psi(f;u)=0$ for $u<0$, this is why we are allowed to "forget" about integrating over $-\infty<u\leqslant 0$. □

The next lemma is taken from Esséen's thesis [9] (see Lemma 1 on page 49).

Lemma 4.13. *Let* $F(t)$ *be a distribution function and denote by* $\phi(t)$ *it's Fourier transform* $\int_{\mathbb{R}} e^{it\,u}dF(u)$. *If* $F(t)$ *is not lattice-distributed, then, for any* $c>0$ *there is a* $\lambda(x)\to\infty$ *and a* $\xi(x)\to\infty$ *such that*

$$\int_c^{\lambda(x)} |\phi(t)|^x\cdot\frac{dt}{t} \ll \frac{1}{\xi(x)\cdot\sqrt{x}}$$

From lemma 4.13 and lemma 4.12 we obtain the following useful estimate.

Lemma 4.14. *Let* $f\in\mathcal{C}$. *Suppose that* $\Psi(f;t)$ *is not lattice distributed. Then, for any* $c>0$ *there is a* $\lambda(x)\to\infty$ *and a* $\xi(x)\to\infty$ *such that uniformly in* $w\geqslant 0$,

$$\int_c^{\lambda(x)} \left|(\log x)^{\widehat{\Psi}(f;w+it)-\widehat{\Psi}(f;w)}\right|\cdot\frac{dt}{t} \ll \frac{1}{\xi(x)\cdot\sqrt{\log\log x}}$$

Proof. Since $\Psi(f;t)$ is not lattice distributed, by lemma 4.12 the function $\phi(t)=e^{\widehat{\Psi}(f;it)-1}$ is the Fourier transform of a non-lattice distribution function. Therefore by lemma 4.13, given any $c>0$ there is a $\lambda(x)\to\infty$ and a $\xi(x)\to\infty$ such that

$$\int_c^{\lambda(x)} \left|e^{\widehat{\Psi}(f;it)-1}\right|^{\log\log x}\cdot\frac{dt}{t} \ll \frac{1}{\xi(x)\cdot\sqrt{\log\log x}} \tag{4.25}$$

By lemma 4.12 we have for all $w\geqslant 0$,

$$\int_c^{\lambda(x)} \left|(\log x)^{\widehat{\Psi}(f;w+it)-\widehat{\Psi}(f;w)}\right|\cdot\frac{dt}{t} \leqslant \int_c^{\lambda(x)} \left|e^{\widehat{\Psi}(f;it)-1}\right|^{\log\log x}\cdot\frac{dt}{t}$$

This together with (4.25) gives the claim. □

We need one more lemma from Esséen's thesis [9] (see theorem 6 on page 27).

Lemma 4.15. *Let* $F(t)$ *be a distribution function. Suppose that* $F(t)$ *is not degenerate (that is* $F(t)$ *does not have a jump of mass 1). Denote by* $\phi(t)$ *the Fourier transform* $\int_{\mathbb{R}} e^{itu}dF(u)$ *of the distribution function* $F(\cdot)$. *There is a* c_0 *and a* c_1 *such that for any interval* I *of size less than* c_0

$$\operatorname*{meas}_{u\in I}\left(|\phi(u)|^2\geqslant 1-\delta\right)\leqslant c_1\cdot\sqrt{\delta}$$

The constants c_1 *and* c_0 *depend at most on the distribution function* F.

Finally, we need one last lemma that will allow us to smooth out $\mathbb{P}(\Omega(f;x) \geqslant t)$ (the smoothing will be negligible because $\Psi(f;t)$ is not lattice distributed). The lemma is essentially what appears in Tenenbaum [18] (first formula in section 3 and first formula in section 4 of his paper).

Lemma 4.16. *Let* $Y(x)$ *be a sequence of random variables. Suppose that each* $Y(x)$ *has an entire moment generating function and define*

$$\Phi_Y(x;t)(z) = \mathbb{E}\left[e^{zY(x)}\right] \cdot e^{-zt}$$

Let $C > 0$ *be given. Then for all* $\kappa > 0$ *and* $M, T > 0$, *we have*

$$\begin{aligned} \mathbb{P}(Y(x) \geqslant t) \quad &= \quad \frac{1}{2\pi i}\int_{\kappa - iM}^{\kappa + iM} \Phi_Y(x;t)(z) \cdot \frac{T dz}{z(z+T)} \\ &\quad + O\left(\mathrm{Err}\right) + O\left(\frac{Te^{\kappa/T}}{M} \cdot \Phi_Y(x;t)(\kappa)\right) \end{aligned} \tag{4.26}$$

and the error term Err is given by

$$\mathrm{Err} = \frac{1}{2\pi i}\int_{\kappa - iM}^{\kappa + iM} \Phi_Y(x;t)(z) \cdot \frac{e^{z/T} \cdot T dz}{(z+T)(z+2T)}$$

Proof. First we establish the above when $M = \infty$. This case follows from the inequalities appearing in Tenenbaum's paper. Let $y^+ = \max(y;0)$. Following Tenenbaum [18] (see first equation in section 3) we have

$$1 - e^{-Ty^+} \quad = \quad \frac{1}{2\pi i}\int_{\kappa - i\infty}^{\kappa + i\infty} e^{zy} \cdot \frac{T dz}{z(z+T)}$$

for all $y \in \mathbb{R}$ and $\kappa, T \geqslant 0$. Let $\chi(\cdot)$ denote the characteristic function of $[0;\infty)$. Again according to Tenenbaum's paper (see beginning of section 4), we have the inequality

$$\begin{aligned} 0 \;\leqslant\; \chi(y) - \left(1 - e^{-Ty^+}\right) \quad &\leqslant \quad \frac{e^2}{e-1} \cdot \left(e^{-T(y+1/T)^+} - e^{-2T(y+1/T)^+}\right) \\ &= \quad \frac{e^2}{e-1} \cdot \frac{1}{2\pi i}\int_{\kappa - i\infty}^{\kappa + i\infty} e^{zy} \cdot \frac{e^{z/T} \cdot T dz}{(z+T)(z+2T)} \end{aligned}$$

It follows that for $u, t \in \mathbb{R}$,

$$\begin{aligned} 0 \quad &\leqslant \quad \chi(u-t) - \frac{1}{2\pi i}\int_{\kappa - i\infty}^{\kappa + i\infty} e^{zu} \cdot e^{-zt} \cdot \frac{T dz}{z(z+T)} \\ &\leqslant \quad \frac{K}{2\pi i}\int_{\kappa - i\infty}^{\kappa + i\infty} e^{zu} \cdot e^{-zt} \cdot \frac{e^{z/T} \cdot T dz}{(z+T)(z+2T)} \end{aligned}$$

with $K = e^2/(e-1)$. Integrating the above inequality over u, with respect to the measure $d\mathbb{P}(Y(x) \leqslant u)$ and applying Fubini's theorem we obtain the claim, in the case $M = \infty$.

To obtain the general case, note that —$\Phi_Y(x;t)(z)| \leqslant \Phi_Y(x;t)(\kappa)$ for $\operatorname{Re} z = \kappa$ and let $\mathcal{R}(\kappa, M) := \{\kappa + it : |t| \geqslant M\}$. By the previous inequality for Φ_Y,

$$\left|\iint_{\mathcal{R}(\kappa,M)} \Phi_Y(x;t)(z) \cdot \frac{Tdz}{z(z+T)}\right| \leqslant \Phi_Y(x;t)(\kappa) \cdot 2\int_M^\infty \frac{Tdt}{t^2} \leqslant \frac{2T}{M} \cdot \Phi_Y(x;t)(\kappa)$$

Therefore

$$\frac{1}{2\pi i}\int_{\kappa - \imath\infty}^{\kappa+\imath\infty} \Phi_Y(x;t)(z)\frac{Tdz}{z(z+T)} = \frac{1}{2\pi i}\int_{\kappa-iM}^{\kappa+iM} \Phi_Y(x;t)(z) \cdot \frac{Tdz}{z(z+T)} + O\left(\Phi_Y(x;t)(\kappa) \cdot \frac{T}{M}\right)$$

We truncate the integral appearing in the term Err in a similar fashion. In this case the truncation contributes $O(Te^{\kappa/T}/M \cdot \Phi_Y(x;t)(\kappa))$. Having truncated our integrals we obtained the "general" case of our lemma. □

We are now in position to prove proposition 4.10.

Proof of Proposition 4.10. Let's keep the notation $\Phi_\Omega(x;t) = \mathbb{E}[e^{z\Omega(f;x)}] \cdot e^{-zt}$ introduced in lemma 4.16 and abbreviate $\mu := \mu(f;x)$, $\sigma := \sigma(f;x)$. Throughout we set $z := \nu + it = \nu_f(x;\Delta) + it$ with $t \in \mathbb{R}$ and we abbreviate $\nu := \nu_f(x;\Delta)$. Note that by lemma 4.7 there is a $C = C(\delta) > 0$ such that $0 \leqslant \nu \leqslant C$ when Δ is in the range $1 \leqslant \Delta \leqslant \delta\sigma(f;x)$. (We allow our error term to depend on C). Also by lemma 4.8, $e^{-\nu(\mu+\Delta\sigma)} \asymp (\log x)^{-\nu\hat{\Psi}'(f;\nu)}$. Thus, by assumptions and this estimate

$$\begin{aligned}\Phi_\Omega(x;\mu+\Delta\sigma)(z) &= \mathcal{A}(z)(\log x)^{\hat{\Psi}(f;z)-1}e^{-z(\mu+\Delta\sigma)} + O((\log x)^{\hat{\Psi}(f;\nu)-3/2}e^{-\nu(\mu+\Delta\sigma)}) \\ &= \mathcal{A}(z)(\log x)^{\hat{\Psi}(f;z)-1} \cdot e^{-z(\mu+\Delta\sigma)} + O\left((\log x)^{A(f;\nu)-1/2}\right) \qquad (4.27)\end{aligned}$$

for $0 \leqslant \nu := \operatorname{Re} z \leqslant C$ and $|\operatorname{Im} z| \leqslant \operatorname{loglog} x$ and where $A(f;\nu) := \hat{\Psi}(f;\nu) - 1 - \nu\hat{\Psi}'(f;\nu)$. We insert (4.27) into (4.26) of the previous lemma. In there we set $\kappa := \nu_f(x;\Delta)$, $Y(x) := \Omega(f;x)$, $M := \operatorname{loglog} x$ and $T := \sqrt{\lambda(x)} \longrightarrow \infty$. The function $\lambda(x)$ is $\ll \operatorname{logloglog} x$ and tends to infinity as $x \to \infty$. It will be specified explicitly later on. We get

$$\begin{aligned}&\mathbb{P}(\Omega(f;x) \geqslant \mu + \Delta\sigma) := \frac{1}{2\pi i}\int_{\nu-iM}^{\nu+iM} \mathcal{A}(z)(\log x)^{\hat{\Psi}(f;z)-1}e^{-z(\mu+\Delta\sigma)} \cdot \frac{Tdz}{z(z+T)} \\ &+O\left(\frac{1}{2\pi i}\int_{\nu-iM}^{\nu+iM} \mathcal{A}(z)(\log x)^{\hat{\Psi}(f;z)-1}e^{-z(\mu+\Delta\sigma)} \cdot \frac{e^{z/T} \cdot Tdz}{(z+T)(z+2T)}\right) \qquad (4.28) \\ &+O\left(\int_{\nu-iM}^{\nu+iM} (\log x)^{A(f;\nu)-1/2} \cdot \frac{e^{\nu/T} \cdot T|dz|}{|z| \cdot |z+T|}\right) + O\left(\frac{T}{M} \cdot \Phi_\Omega(x;\mu+\Delta\sigma)(\nu)\right)\end{aligned}$$

At the outset note that the very last error term is negligible. Indeed, by (4.27) and the boundedness of $\mathcal{A}(\nu)$ in $0 \leqslant \nu \leqslant C$ (the function $\mathcal{A}(\cdot)$ is continuous!), we have $\Phi_\Omega(x;\mu + \Delta\sigma)(\nu) \ll (\log x)^{A(f;\nu)}$. Therefore $T/M \cdot \Phi_\Omega(x;\mu+\Delta\sigma)(\nu) \ll (\operatorname{logloglog} x/\operatorname{loglog} x) \cdot (\log x)^{A(f;\nu)}$ which is negligible compared to the expected size of the main term.

The integral over $T \cdot |dz|/|z||z+T|$ contributes less than $\nu^{-1} + T \ll \nu^{-1}(1+T)$. Thus the second error term in (4.28) is $\ll (\log x)^{A(f;\nu)-1/2}\nu^{-1}(1+T)$. Since $T \ll \log\log\log x$ this error term is negligible compared to the expected size of the main term.

Once we evaluate the main term in (4.28) it will be clear how to bound the first error term in (4.28). Therefore let's focus on estimating

$$\frac{1}{2\pi i}\int_{\nu-iM}^{\nu+iM} \mathcal{A}(z)(\log x)^{\widehat{\Psi}(f;z)-1}e^{-z(\mu+\Delta\sigma)} \cdot \frac{T dz}{z\cdot(z+T)} \tag{4.29}$$

This corresponds to the main term for $\mathbb{P}(\Omega(f;x) \geqslant \mu(f;x) + \Delta\sigma(f;x))$. We split (4.29) into a part over $\mathcal{M} := \{\nu + it : |t| \leqslant \eta(x)\cdot(\log\log x)^{-1/2}\}$ where $\eta(x) = \log\log\log x$ and a part over $\mathcal{R} = \{\nu + it : |t| \leqslant M\} - \mathcal{M}$. The part over $\mathcal{M}$ will furnish the main term and the part over $\mathcal{R}$ will be negligible.

1. Asymptotic for (4.29) restricted to $z = \nu + it \in \mathcal{M}$.

By lemma 4.8 for $z \in \mathcal{M} = \left\{\nu + it : |t| \leqslant \eta(x)\cdot(\log\log x)^{-1/2}\right\}$,

$$e^{-z(\mu+\Delta\sigma)} = (\log x)^{-z\widehat{\Psi}'(f;\nu)} \cdot e^{-zc(f)} \cdot (1 + O((\log\log x)^{-1/2}))$$

Therefore, for $z \in \mathcal{M}$,

$$\begin{aligned} & \mathcal{A}(z)(\log x)^{\widehat{\Psi}(f;z)-1} \cdot e^{-z(\mu+\Delta\sigma)} \\ = \; & \mathcal{A}(z)e^{-zc(f)} \cdot (\log x)^{\widehat{\Psi}(f;z)-1-z\widehat{\Psi}'(f;\nu)} \cdot \left(1 + O\left((\log\log x)^{-1/2}\right)\right) \\ = \; & \mathcal{A}(z)e^{-zc(f)} \cdot (\log x)^{\widehat{\Psi}(f;z)-1-z\widehat{\Psi}'(f;\nu)} + O\left((\log x)^{A(f;\nu)} \cdot (\log\log x)^{-1/2}\right) \end{aligned}$$

In the third line we use the fact that $|\widehat{\Psi}(f;z)| \leqslant \widehat{\Psi}(f;\nu)$ and that $\mathcal{A}(z)$ is analytic hence bounded in the (bounded) region $0 \leqslant \nu := \operatorname{Re} z \leqslant C$, $|\operatorname{Im} z| \leqslant 2$. Plugging the above estimate into (4.29) (restricted to $z \in \mathcal{M}$) yields

$$\begin{aligned} & \frac{1}{2\pi i}\int_{\mathcal{M}} \mathcal{A}(z)(\log x)^{\widehat{\Psi}(f;z)-1} \cdot e^{-z(\mu+\Delta\sigma)} \cdot \frac{T dz}{z(z+T)} \\ = \; & \frac{1}{2\pi i}\int_{\mathcal{M}} \mathcal{A}(z)e^{-zc(f)} \cdot (\log x)^{\widehat{\Psi}(f;z)-1-z\widehat{\Psi}'(f;\nu)} \cdot \frac{T dz}{z(z+T)} \\ & + O\left(\frac{(\log x)^{A(f;\nu)}}{\sqrt{\log\log x}}\int_{\mathcal{M}} \frac{T\cdot|dz|}{|z|\cdot|z+T|}\right) \end{aligned} \tag{4.30}$$

and the error term is bounded by $O((\log x)^{A(f;\nu)} \cdot \nu^{-1} \cdot \eta(x)(\log\log x)^{-1})$ (note that $\mathcal{M}$ is in length $\ll \eta(x)/(\log\log x)^{1/2}$) which is negligible when compared to the expected size of the main term. We parametrize the integral in (4.30) and perform a series of Taylor expansions. Recall that $z := \nu + it$ by convention, that $0 \leqslant \nu = \nu_f(x;\Delta) \leqslant C$ whenever $1 \leqslant \Delta \leqslant \delta\sigma(f;x)$ and that when $z \in \mathcal{M}$ then $|t| \leqslant \eta(x)\cdot(\log\log x)^{-1/2}$. With this in mind, for $z = \nu + it \in \mathcal{M}$,

$$\begin{aligned}\mathcal{A}(z)e^{-zc(f)} &= \mathcal{A}(\nu)e^{-\nu c(f)} + O_C(|t|) = \mathcal{A}(\nu)e^{-\nu c(f)} + O_C\left(\eta(x)\cdot(\log\log x)^{-1/2}\right)\\ &= \mathcal{A}(\nu)e^{-\nu c(f)}\cdot\left(1 + O_C\left(\eta(x)(\log\log x)^{-1/2}\right)\right)\end{aligned}$$

where the last line is justified by 1) the non-vanishing of $\mathcal{A}(x)e^{-xc(f)}$ on the positive real axis 2) the fact that $\nu \asymp \Delta/\sigma(f;x)$ is bounded throughout $1 \leqslant \Delta \leqslant c\sigma(f;x)$ $(0 \leqslant \nu \leqslant C)$. We will not mention any further, the dependence on C in implicit constants. Proceeding as in the previous equation, we find

$$\widehat{\Psi}(f;z) - \widehat{\Psi}(f;\nu) - it\,\widehat{\Psi}'(f;\nu) = -\left(t^2/2\right)\widehat{\Psi}''(f;\nu) + O\left(\eta(x)^3\cdot(\log\log x)^{-3/2}\right)$$

for $z = \nu + it \in \mathcal{M}$. Upon multiplying by $\log\log x$ and exponentiating, we obtain

$$(\log x)^{\widehat{\Psi}(f;z)-\widehat{\Psi}(f;\nu)-it\,\widehat{\Psi}'(f;\nu)} = e^{-(t^2/2)\widehat{\Psi}''(f;\nu)\log\log x}\cdot\left(1 + O\left(\eta(x)^3(\log\log x)^{-1/2}\right)\right)$$

By lemma 4.7 for $z = \nu + it \in \mathcal{M}$ we have $|t/\nu| \ll \eta(x)(\log\log x)^{-1/2}\cdot\nu^{-1} \asymp \eta(x)\cdot\Delta^{-1}$. Since $\eta(x) = \log\log\log x = o(\Delta)$ it follows that $|t/\nu| = o(1)$ when $\nu + it \in \mathcal{M}$. Since in addition we will choose $T \to \infty$, we have for $z = \nu + it \in \mathcal{M}$,

$$\begin{aligned}\frac{T}{z\cdot(z+T)} &= \frac{1}{\nu+it}\cdot\frac{1}{1+(\nu+it)/T}\\ &= \frac{1}{\nu}\cdot(1 + O(t/\nu))\cdot(1 + O(1/T))\\ &= \frac{1}{\nu}\cdot\left(1 + O\left(1/T + \eta(x)(\log\log x)^{-1/2}\cdot\nu^{-1}\right)\right)\end{aligned}$$

Collecting together the previous estimates, we conclude that for $z = \nu + it \in \mathcal{M}$,

$$\begin{aligned}&\mathcal{A}(z)e^{-zc(f)}\cdot(\log x)^{\widehat{\Psi}(f;z)-\widehat{\Psi}(f;\nu)-it\,\widehat{\Psi}'(f;\nu)}\cdot\frac{T}{z(z+T)}\\ =\ &\left(\mathcal{A}(\nu)e^{-\nu c(f)}/\nu\right)\cdot e^{-(t^2/2)\widehat{\Psi}''(f;\nu)\log\log x}\cdot\left(1 + O\left(1/T + \eta(x)^3\cdot(\log\log x)^{-1/2}\cdot\nu^{-1}\right)\right)\end{aligned}$$

Since $\nu \asymp \Delta\cdot(\log\log x)^{-1/2}$ the error term simplifies to $\mathcal{E} := 1/T + \eta(x)^3\cdot\Delta^{-1}$. Let $\xi := \eta(x)\cdot(\log\log x)^{-1/2}$. Parametrizing the integral in (4.30) and using the previous asymptotic we find that the integral in (4.30) equals (where as usual $z := \nu + it$)

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\xi}^{\xi} \mathcal{A}(z)e^{-zc(f)} \cdot (\log x)^{\widehat{\Psi}(f;z)-1-z\widehat{\Psi}'(f;\nu)} \cdot \frac{T dt}{z(z+T)} \\
= \ & (\log x)^{A(f;\nu)} \cdot \frac{1}{2\pi}\int_{-\xi}^{\xi} \mathcal{A}(z)e^{-zc(f)} \cdot (\log x)^{\widehat{\Psi}(f;z)-\widehat{\Psi}(f;\nu)-it\widehat{\Psi}'(f;\nu)} \cdot \frac{T dt}{z(z+T)} \\
= \ & (\log x)^{A(f;\nu)} \cdot (1/\nu)\mathcal{A}(\nu)e^{-\nu c(f)} \int_{-\xi}^{\xi} e^{-(t^2/2)\widehat{\Psi}''(f;\nu)\cdot \log\log x} \cdot \frac{dt}{2\pi} \cdot (1 + O(\mathcal{E})) \\
= \ & (\log x)^{A(f;\nu)} \cdot \frac{\mathcal{A}(\nu)e^{-\nu c(f)}}{\nu(\widehat{\Psi}''(f;\nu)\log\log x)^{1/2}} \int_{-\eta(x)}^{\eta(x)} e^{-u^2/2} \cdot \frac{du}{2\pi} \cdot (1 + O(\mathcal{E})) \\
= \ & (\log x)^{A(f;\nu)} \cdot \frac{\mathcal{A}(\nu)e^{-\nu c(f)}}{\nu(2\pi\widehat{\Psi}''(f;\nu)\log\log x)^{1/2}} \cdot \left(1 + O\left(e^{-\eta(x)^2/2} + \mathcal{E}\right)\right)
\end{aligned}$$

Since $\mathcal{E} := 1/T + \eta(x)^3/\Delta \gg 1/\sqrt{\log\log x}$ and $\eta(x) \gg \log\log\log x$ the term $e^{-\eta^2/2}$ is absorbed into $O(\mathcal{E})$. The integral we just evaluated furnishes the main term in (4.30). We conclude that

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{\mathcal{M}} \mathcal{A}(z)(\log x)^{\widehat{\Psi}(f;z)-1} \cdot e^{-z(\mu+\Delta\sigma)} \cdot \frac{T dz}{z(z+T)} \\
= \ & \mathcal{A}(\nu)e^{-\nu c(f)} \cdot \frac{(\log x)^{\widehat{\Psi}(f;\nu)-1-\nu\widehat{\Psi}'(f;\nu)}}{\nu(2\pi\widehat{\Psi}''(f;\nu)\log\log x)^{1/2}} \cdot \left(1 + O\left(\frac{1}{T} + \frac{\eta(x)^3}{\Delta}\right)\right)
\end{aligned} \tag{4.31}$$

2. Bound for (4.29) restricted to $z = \nu + it \in \mathcal{R}$.

Recall our convention that $z := \nu + it$ and $\nu := \nu_f(x;\Delta)$. By lemma 4.8, and the bound $\mathcal{A}(z) \ll 1 + |z|^{1/8}$ we have

$$\mathcal{A}(z)(\log x)^{\widehat{\Psi}(f;z)-1} \cdot e^{-z(\mu+\Delta\sigma)} \ll (1 + |z|^{1/8}) \cdot (\log x)^{\mathrm{Re}(\widehat{\Psi}(f;z)-1-\nu\widehat{\Psi}'(f;\nu))}$$

uniformly in $0 \leqslant \nu := \mathrm{Re}\, z \leqslant C$ and $|\mathrm{Im}\, z| \leqslant M := \log\log x$. Therefore

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{\mathcal{R}} \mathcal{A}(z)(\log x)^{\widehat{\Psi}(f;z)-1} \cdot e^{-z(\mu+\Delta\sigma)} \cdot \frac{T dz}{z(z+T)} \\
\ll \ & (\log x)^{A(f;\nu)} \cdot \int_{\mathcal{R}} (1 + |z|^{1/8}) \cdot (\log x)^{\mathrm{Re}(\widehat{\Psi}(f;z)-\widehat{\Psi}(f;\nu))} \cdot \frac{T \cdot |dz|}{|z| \cdot |z+T|}
\end{aligned} \tag{4.32}$$

and it remains to bound the second integral, above. To do so we consider its behaviour in three ranges, $\mathcal{R}_1 = \{\nu + it : \eta(x) \cdot (\log\log x)^{-1/2} \leqslant |t| \leqslant c\}$ with $c > 0$ small enough, $\mathcal{R}_2 = \{\nu + it : c \leqslant |t| \leqslant \lambda(x)\}$ and $\mathcal{R}_3 = \{\nu + it : \lambda(x) \leqslant |t| \leqslant M\}$. We will fix c and $\lambda(x)$ as we proceed through the proof. Recall also that $T = \lambda(x)^{1/2}$ and that $\eta(x) = \log\log\log x$.

2.1. The range $\mathcal{R}_1 = \{\nu + it : \eta(x) \cdot (\log\log x)^{-1/2} \leqslant |t| \leqslant c\}$

Recall from lemma 4.12 that $\mathrm{Re}(\widehat{\Psi}(f;z)-\widehat{\Psi}(f;\nu)) \leqslant \mathrm{Re}(\widehat{\Psi}(f;it)-1)$. Choose $c \leqslant 1$ small enough so as to ensure that for $z = \nu + it$,

$$\mathrm{Re}(\widehat{\Psi}(f;z)-\widehat{\Psi}(f;\nu)) \leqslant \mathrm{Re}(\widehat{\Psi}(f;it)-1) \leqslant -\kappa t^2/2$$

for some $\kappa = \kappa(c) > 0$. The existence of such a κ, for c small enough, is guaranteed by a Taylor expansion. Once $c \leqslant 1$ is chosen sufficiently small, we can bound

$$\begin{aligned} & \int_{\mathcal{R}_1} (1+|z|^{1/8}) \cdot (\log x)^{\mathrm{Re}(\widehat{\Psi}(f;z)-\widehat{\Psi}(f;\nu))} \cdot \frac{T|dz|}{|z|\cdot|z+T|} \\ \ll \; & \int_{\mathcal{R}_1} (\log x)^{-\kappa t^2/2} \cdot \frac{Tdt}{\nu(\nu+T)} \; \ll \; (1/\nu)\exp(-\kappa\eta(x)^2/2) \end{aligned}$$

The last line comes from $|t| \geqslant \eta(x) \cdot (\log\log x)^{-1/2}$. Since $\eta(x) = \log\log\log x$ it follows that (4.32) restricted to the range $\mathcal{R}_1$ is $\ll (\log x)^{A(f;\nu)} \cdot (1/\nu)(\log\log x)^{-1}$ and this is as negligible as we want it to be.

2.2. The range $\mathcal{R}_2 = \{\nu + it : c \leqslant |t| \leqslant \lambda(x)\}$

Let $c > 0$ denote the constant that we fixed in the previous point. By lemma 4.14 there is a $\lambda_0(x) \to \infty$ and a $\xi(x) \to \infty$ such that

$$\int_c^{\lambda_0(x)} \left| e^{\widehat{\Psi}(f;z)-\widehat{\Psi}(f;\nu)} \right|^{\log\log x} \cdot \frac{dt}{t} \ll \frac{1}{\xi(x)\sqrt{\log\log x}}\, , z := \nu + it$$

Let $\lambda(x) := \min(\lambda_0(x), \xi(x), 1 + \log\log\log x)$. Note that $\lambda(x) \to \infty$. By the above equation and $\lambda(x) \leqslant \lambda_0(x)$ we get

$$\int_c^{\lambda(x)} \left| e^{\widehat{\Psi}(f;z)-\widehat{\Psi}(f;\nu)} \right|^{\log\log x} \cdot \frac{dt}{t} \ll \frac{1}{\xi(x)\sqrt{\log\log x}} \tag{4.33}$$

We let $T := \sqrt{\lambda(x)}$. With this choice of T, we have $T \to \infty$ and $T \ll \log\log\log x$, and this is the only information about T that we assumed *a priori*. By (4.33) we have

$$\begin{aligned} & \int_{\mathcal{R}_2} (1+|z|^{1/8}) \cdot (\log x)^{\mathrm{Re}(\widehat{\Psi}(f;z)-\widehat{\Psi}(f;\nu))} \cdot \frac{T|dz|}{|z|\cdot|z+T|} \\ \ll \; & \lambda(x)^{1/8} \cdot \int_{\mathcal{R}_2} \left| e^{\widehat{\Psi}(f;z)-\widehat{\Psi}(f;\nu)} \right|^{\log\log x} \cdot \frac{dt}{t} \; \ll \; \frac{\lambda(x)^{1/8}}{\xi(x)\sqrt{\log\log x}} \end{aligned}$$

Since $\lambda(x) \leqslant \xi(x)$ the above is bounded by $\xi(x)^{-7/8} \cdot (\log\log x)^{-1/2}$. It follows that (4.32) resitricted to $z \in \mathcal{R}_2$ is bounded by $(\log x)^{A(f;\nu)} \cdot \xi(x)^{-7/8} \cdot (\log\log x)^{-1/2}$ and again this is sufficiently negligible, for our purpose.

2.3. The range $\mathcal{R}_3 := \{\nu + it : \lambda(x) \leqslant |t| \leqslant M\}$.

Since $0 \leqslant \nu \leqslant C$ and $z = \nu + it$ we have $|z| \ll |t|$. It follows that

$$\begin{aligned}
& \int_{\mathcal{R}_3} (1 + |z|^{1/8}) \cdot (\log x)^{\mathrm{Re}(\hat{\Psi}(f;z) - \hat{\Psi}(f;\nu))} \cdot \frac{T \cdot |dz|}{|z| \cdot |z+T|} \\
\ll \ & \int_{\lambda(x)}^{M} t^{1/8} \cdot \left| e^{\hat{\Psi}(f;z) - \hat{\Psi}(f;\nu)} \right|^{\log\log x} \cdot \frac{T dt}{t \cdot (t+T)} \\
\ll \ & \sum_{\ell \geqslant \lfloor \lambda(x) \rfloor} \frac{T}{\ell^{7/8} \cdot (\ell + T)} \int_{\ell}^{\ell+1} \left| e^{\hat{\Psi}(f;z) - \hat{\Psi}(f;\nu)} \right|^{\log\log x} dt \qquad (4.34)
\end{aligned}$$

We now show that the integral over $\ell \leqslant t \leqslant \ell + 1$ is $\ll (\log\log x)^{-1/2}$ uniformly in $\ell \geqslant 0$. Let $\phi_\kappa(t) := |e^{\hat{\Psi}(f;\kappa+it) - \hat{\Psi}(f;\kappa)}|$ and $\xi := \log\log x$, so $|\phi_\nu(t)|^\xi = |e^{\hat{\Psi}(f;z) - \hat{\Psi}(f;\nu)}|^{\log\log x}$ ($\nu := \mathrm{Re}\, z = \nu_f(x;\Delta)$). By lemma 4.12 we have $|\phi_\kappa(t)| \leqslant |\phi_0(t)|$ for all $\kappa > 0$ and $t \in \mathbb{R}$. Furthermore by lemma 4.15 there is a c_0 and a c_1 such that $\mathrm{meas}(\{u \in I : |\phi_0(u)|^2 \geqslant 1 - \delta\}) \leqslant c_1 \cdot \sqrt{\delta}$ for all intervals I of length $\leqslant c_0$. In particular $\mathrm{meas}(\{u \in [\ell; \ell+1] : |\phi_0(u)|^2 \geqslant 1 - \delta\}) \leqslant K \cdot \sqrt{\delta}$ where $K := (1/c_0 + 1) \cdot c_1$. Using these two observations we conclude that

$$\begin{aligned}
\int_{\ell}^{\ell+1} |\phi_\nu(t)|^\xi \cdot dt \ & \leqslant \ \int_{\ell}^{\ell+1} |\phi_0(t)|^\xi \cdot dt \ = - \int_0^1 t^{\xi/2} \cdot d\left(\operatorname*{meas}_{u \in [\ell;\ell+1]} \left(|\phi_0(u)|^2 \geqslant t \right) \right) \\
& = \ (\xi/2) \cdot \int_0^1 t^{\xi/2 - 1} \cdot \operatorname*{meas}_{u \in [\ell;\ell+1]} \left(|\phi_0(u)|^2 \geqslant t \right) dt \\
& \leqslant \ (\xi/2) \cdot K \int_0^1 t^{\xi/2-1} \cdot \sqrt{1-t} dt \ \ll \ \xi^{-1/2}
\end{aligned}$$

We evaluate the last integral by noticing that the integrand is essentially constant on intervals $[1 - (A+1)/\xi; 1 - A/\xi]$. The long chain of inequalities proves that

$$\int_{\ell}^{\ell+1} \left| e^{\hat{\Psi}(f;z) - \hat{\Psi}(f;\nu)} \right|^{\log\log x} \cdot dt = \int_{\ell}^{\ell+1} |\phi_\nu(t)|^\xi dt \ll \xi^{-1/2} = (\log\log x)^{-1/2}$$

as desired. Hence the sum in (4.34) is bounded by

$$\ll \ \frac{1}{\sqrt{\log\log x}} \sum_{\ell \geqslant \lfloor \lambda(x) \rfloor} \frac{T}{\ell^{7/8} \cdot (\ell + T)} \ \ll \ \frac{1}{\lambda(x)^{3/8}} \cdot \frac{1}{\sqrt{\log\log x}}$$

(Recall that $T = \lambda(x)^{1/2}$). It follows that the integral in (4.32) restricted to $z \in \mathcal{R}_3$ is bounded by $(\log x)^{A(f;\nu)} \cdot (\log\log x)^{-1/2} \cdot \lambda(x)^{-3/8}$, which is sufficiently negligible for our purpose.

2.4. Final bound for (4.32).

Collecting the previous bounds from 2.1, 2.2, and 2.3, we conclude that

$$\int_{\mathcal{R}} \mathcal{A}(z)(\log x)^{\widehat{\Psi}(f;z)-1} \cdot e^{-z(\mu+\Delta\sigma)} \cdot \frac{T \cdot dz}{z(z+T)}$$
$$\ll \quad (\log x)^{A(f;\nu)} \cdot \left(\frac{1}{\nu \log\log x} + \frac{\xi(x)^{-7/8} + \lambda(x)^{-3/8}}{(\log\log x)^{1/2}} \right)$$

which is negligible compared to the estimate we obtained in (4.31), because $\mathcal{A}(\nu)e^{-\nu c(f)} \asymp 1$ and $\widehat{\Psi}''(f;\nu) \asymp 1$. The estimate $\mathcal{A}(\nu)e^{-\nu c(f)} \asymp 1$ follows from the continuity and non-vanishing of $\mathcal{A}(x)$ on the positive real line, and the fact that the parameter ν is confined to a bounded interval $0 \leqslant \nu \leqslant C$.

3. Conclusion.

Comparing the bound we obtained in 2.4 with (4.31) it follows that

$$\frac{1}{2\pi i} \int_{\nu-iM}^{\nu+iM} \mathcal{A}(z)(\log x)^{\widehat{\Psi}(f;z)-1} e^{-z(\mu+\Delta\sigma)} \cdot \frac{T dz}{z(z+T)} \tag{4.35}$$
$$= \quad \mathcal{A}(\nu)e^{-\nu c(f)} \cdot \frac{(\log x)^{\widehat{\Psi}(f;\nu)-1-\nu\widehat{\Psi}'(f;\nu)}}{\nu(2\pi\widehat{\Psi}''(f;\nu)\log\log x)^{1/2}} \cdot (1+o(1))$$

uniformly throughout $1 \leqslant \Delta \leqslant \delta\sigma(f;x)$. In the same way as we estimated the above integral we estimate the integral appearing in the first error term in (4.28). Because of the additional $z+T$ in the denominator this integral will be negligible compared to (4.35). Since the other error terms in (4.28) are negligible compared to (4.35) we finally conclude that

$$\mathbb{P}\left(\Omega(f;x) \geqslant \mu + \Delta\sigma\right) \sim \mathcal{A}(\nu)e^{-\nu c(f)} \cdot \frac{(\log x)^{\widehat{\Psi}(f;\nu)-1-\nu\widehat{\Psi}'(f;\nu)}}{\nu(2\pi\widehat{\Psi}''(f;\nu)\log\log x)^{1/2}}$$

uniformly in $1 \leqslant \Delta \leqslant \delta\sigma(f;x)$, as desired. $\square$

4.5. **Large deviations:** $(\log\log x)^{\varepsilon} \ll \Delta \ll \sigma(f;x)$ **and** $\Psi(f;t)$ **is lattice distributed.** We may assume by rescaling that $\Psi(f;t)$ is lattice distributed on $\mathbb{Z}$. Throughout this section we write $f = \mathfrak{g} + \mathfrak{h}$ with $\mathfrak{g}, \mathfrak{h}$ two strongly additive functions defined by

$$\mathfrak{g}(p) = \begin{cases} f(p) \text{ if } f(p) \in \mathbb{Z} \\ 0 \quad \text{otherwise} \end{cases} \quad \text{and} \quad \mathfrak{h}(p) = \begin{cases} f(p) \text{ if } f(p) \notin \mathbb{Z} \\ 0 \quad \text{otherwise} \end{cases}$$

The goal is to prove the following "general" proposition.

Proposition 4.17. *Let $f \in \mathcal{C}$. Suppose that $\Psi(f;t)$ is lattice distributed on $\mathbb{Z}$. Consider the random variable $\Omega(f;x) := \sum_{p \leqslant x} f(p) Z_p$ where the $Z_p \in \{0;1\}$ are random variables, not necessarily independent, over a common probability space $(\Omega_x, \mathcal{F}_x, \mathbb{P}_x)$ which we allow to depend on x. We denote by $\mathbb{P}_{\mathcal{F}_x}$ and $\mathbb{E}_{\mathcal{F}_x}$ the probability measure and the expectation in that probability space. Suppose that*

1. *Uniformly in* $0 \leqslant \kappa := \operatorname{Re} s \leqslant C, |\operatorname{Im} s| \leqslant \log\log x$ *and uniformly in strongly additive function* $\mathfrak{H}$ *such that* $0 \leqslant \mathfrak{H}(p) \leqslant \lceil \mathfrak{h}(p) \rceil$,

$$\mathbb{E}_{\mathcal{F}_x}\left[e^{s\Omega(\mathfrak{g};x)+s\Omega(\mathfrak{H};x)}\right] = \mathbb{E}_{\mathcal{F}_x}\left[e^{s\Omega(\mathfrak{g};x)}\right] \cdot \prod_{p \leqslant x}\left(1 + \frac{e^{s\mathfrak{H}(p)} - 1}{p}\right) + O_C\left(\mathcal{E}(x;\kappa)\right)$$

with an error term $\mathcal{E}(x;\kappa) := (\log x)^{\hat{\Psi}(f;\kappa)-3/2}$.

2. *Given* $C > 0$, *we have, uniformly in* $0 \leqslant \kappa := \operatorname{Re} s \leqslant C$, $|\operatorname{Im} s| \leqslant 2\pi$,

$$\mathbb{E}_{\mathcal{F}_x}\left[e^{s\Omega(\mathfrak{g};x)}\right] = \mathcal{A}(s) \cdot (\log x)^{\hat{\Psi}(f;s)-1} + O\left((\log x)^{\hat{\Psi}(f;\kappa)-3/2}\right)$$

where $\mathcal{A}(s)$ *is an analytic function in* $\operatorname{Re} s \geqslant 0$, *which we assume to be non-zero on the positive real axis.*

Then, for any given $c > 0$, *uniformly in* $1 \leqslant \Delta \leqslant c\sigma(f;x)$,

$$\mathbb{P}_{\mathcal{F}_x}\left(\sum_{p \leqslant x} f(p)\left[Z_p - \frac{1}{p}\right] \geqslant \Delta\sigma(f;x)\right) \sim \mathcal{A}(\nu)e^{-\nu c(f)} \cdot \mathcal{P}_{\mathfrak{h}}\left(\xi_f(x;\Delta);\nu\right) \cdot S_f(x;\Delta)$$

with $\nu := \nu_f(x;\Delta)$ *and the rest of the notation defined in the table of section 3.*

In the most important case, when $\Omega_x = [1;x]$ and the random variables $Z_p(n)$ are the indicator functions of the event $p|n$, proposition 4.17 can be proved by following the method of [1]. The proof there is more natural, but unfortunately doesn't adapt to a more general situation, in particular to the case when the Z_p are independent random variables.

We need a substantial amount of preparation before we can prove the lemma. We subdivide this section in three subsections. In 4.5.1 we gather information about the additive function $\mathfrak{h}$. In 4.5.2 we evaluate a certain "saddle-point" integral. In 4.5.3 we prove proposition 4.17.

4.5.1. *Preliminary lemma on* $\mathfrak{h}$. Denote by $S(\mathfrak{h})$ the set of primes for which $\mathfrak{h}(p) \neq 0$. Recall that $\mathfrak{h}(p)$ is equal to $f(p)$ whenever $f(p) \notin \mathbb{Z}$ and equal to 0 otherwise. Since $f(p) > 0$ (by definition of the class $\mathcal{C}$) it follows that $\mathfrak{h}(p)$ vanishes exactly when $f(p) \in \mathbb{Z}$. Hence the set $S(\mathfrak{h}) = \{p : \mathfrak{h}(p) \neq 0\}$ is in fact equal to the set $\{p : f(p) \notin \mathbb{Z}\}$.

Lemma 4.18. *Let* $f \in \mathcal{C}$. *Suppose that* $\Psi(f;t)$ *is lattice distributed on* $\mathbb{Z}$. *We have*

$$|S(\mathfrak{h}) \cap [1;x]| \ll_A x \cdot (\log x)^{-A}$$

Proof. By our remark above $S(\mathfrak{h}) = \{p : f(p) \notin \mathbb{Z}\}$. By assumptions (1.4) we have for arbitrary $a \in \mathbb{Z}$

$$\frac{1}{\pi(x)} \sum_{\substack{p \leqslant x \\ f(p) \leqslant a}} 1 = \Psi(f;a) + O_A\left((\log x)^{-A-1}\right) \tag{4.36}$$

Further since $\Psi(f;t)$ is a distribution function it is right continuous, so

$$\frac{1}{\pi(x)} \sum_{\substack{p \leqslant x \\ f(p) < a+1}} 1 = \lim_{t \uparrow a+1} \Psi(f;t) + O_A\left((\log x)^{-A-1}\right) \tag{4.37}$$

Since $\Psi(f;t)$ is lattice distributed on $\mathbb{Z}$ it is constant on the interval $[a;a+1)$. Therefore the right hand side of (4.36) and (4.37) are equal. Hence subtracting (4.36) from (4.37) yields

$$\frac{1}{\pi(x)} \sum_{\substack{p \leqslant x \\ a < f(p) < a+1}} 1 = O_A\left(\frac{1}{(\log x)^{A+1}}\right) \tag{4.38}$$

uniformly in $a \in \mathbb{Z}$. By assumption (1.3) there are only $O(1)$ primes $p \leqslant x$ such that $f(p) \geqslant \log x$. Therefore

$$\frac{1}{\pi(x)} \sum_{\substack{p \leqslant x \\ f(p) \notin \mathbb{Z}}} 1 \leqslant \sum_{0 \leqslant a \leqslant \log x} \frac{1}{\pi(x)} \sum_{\substack{p \leqslant x \\ a < f(p) < a+1}} 1 + \frac{1}{\pi(x)} \sum_{\substack{p \leqslant x \\ f(p) \geqslant \log x}} 1$$

by (4.38) the above sum is $O_A(\log x \cdot (\log x)^{-A-1})$ as desired. □

As a consequence of the lemma $\prod_{p \in S(\mathfrak{h})}(1 + 1/p)$ converges and $\Psi(f;t) = \Psi(\mathfrak{g};t)$. In fact we proved a little bit more.

Corollary. *Let $f \in \mathcal{C}$. Suppose that $\Psi(f;t)$ is lattice distributed on $\mathbb{Z}$. Let $A > 0$ be given. The sum*

$$\sum_{p | n \Rightarrow p \in S(\mathfrak{h})} \frac{(\log n)^A}{n}$$

converges.

Proof. For an integer n with prime factorization $n = p_1^{\alpha_1} \cdot \ldots \cdot p_k^{\alpha_k}$ we have the inequality $\log n \leqslant \prod_{\ell \leqslant k}(\alpha_\ell \cdot \log p_\ell + 1)$. Therefore

$$\sum_{p | n \Rightarrow p \in S(\mathfrak{h})} \frac{(\log n)^A}{n} \leqslant \prod_{p \in S(\mathfrak{h})} \left(1 + \sum_{\alpha \geqslant 1} \frac{(\alpha \log p + 1)^A}{p^\alpha}\right) \ll \prod_{p \in S(\mathfrak{h})} \left(1 + K \cdot \frac{(\log p)^A}{p}\right)$$

for some constant $K > 0$. By lemma 4.18, $\sum_{p \in S(\mathfrak{h})} (\log p)^A \cdot p^{-1} < +\infty$ therefore the product on the right is finite. □

We need more than mere convergence of the product $\prod_{p \in S(\mathfrak{h})}(1 - 1/p)$.

Lemma 4.19. *Let $f \in \mathcal{C}$. Suppose that $\Psi(f;t)$ is lattice distributed on $\mathbb{Z}$. The function*

$$G(\mathfrak{h};s) := \prod_{p \in S(\mathfrak{h})} \left(1 + \frac{e^{s\mathfrak{h}(p)}}{p-1}\right) \cdot \left(1 - \frac{1}{p}\right)$$

is entire and the product converges for all $s \in \mathbb{C}$. *Furthermore, given* $\delta > 0$, *there is a* $x_0(\delta)$ *such that uniformly in* $\operatorname{Re} s \leqslant \delta$ *and* $x \geqslant x_0(\delta)$,

$$\prod_{\substack{p \leqslant x \\ p \in S(\mathfrak{h})}} \left(1 + \frac{e^{s\mathfrak{h}(p)}}{p-1}\right) \cdot \left(1 - \frac{1}{p}\right) = G(\mathfrak{h}; s) \cdot \left(1 + O_\delta\left((\log x)^{-1/2}\right)\right) \tag{4.39}$$

Remark. Note that $G(\mathfrak{h}; s)$ is the moment generating function of the random variable $X(\mathfrak{h}) := \sum_p \mathfrak{h}(p) X_p$. Indeed,

$$\mathbb{E}\left[e^{sX(\mathfrak{h})}\right] = \prod_p \left(1 - \frac{1}{p} + \frac{e^{s\mathfrak{h}(p)}}{p}\right) = \prod_{p \in S(\mathfrak{h})} \left(1 + \frac{e^{s\mathfrak{h}(p)}}{p-1}\right) \cdot \left(1 - \frac{1}{p}\right)$$

In particular $\mathbb{E}\left[e^{\kappa X(\mathfrak{h})}\right]$ is finite for any fixed $\kappa > 0$.

Proof. We are going to show that $\sum_{p > x, \mathfrak{h}(p) \neq 0} \log\left(1 + (e^{s\mathfrak{h}(p)} - 1)/p\right) \ll (\log x)^{-1/2}$ uniformly in $\operatorname{Re} s \leqslant \delta$, for all x large enough (we need to take x large enough to prevent $1 + (e^{s\mathfrak{h}(p)} - 1)/p$ from vanishing when $\operatorname{Re} s \leqslant \delta$ and $p > x$). This bound admits two consequences. First of all, it implies that the partial products

$$\prod_{p \leqslant x} \left(1 + \frac{e^{s\mathfrak{h}(p)}}{p-1}\right) \cdot \left(1 - \frac{1}{p}\right) = \prod_{\substack{p \leqslant x \\ \mathfrak{h}(p) \neq 0}} \left(1 + \frac{e^{s\mathfrak{h}(p)} - 1}{p}\right) \tag{4.40}$$

converge uniformly on compact subsets of $\mathbb{C}$. Hence $G(\mathfrak{h}; s)$ is an entire function. Secondly, since (4.40) converges to $G(\mathfrak{h}; s)$ and its tails are $1 + O((\log x)^{-1/2})$ we obtain (4.39). Thus it remains to bound the sum of $\log(1 + (e^{s\mathfrak{h}(p)} - 1)/p)$ over $p > x$. Assume without loss of generality that $\delta \geqslant 2$. By assumption (1.3), $f(p) = o(\log p)$. In particular $\mathfrak{h}(p) = o(\log p)$ and thus $|e^{s\mathfrak{h}(p)}| \leqslant e^{\delta\mathfrak{h}(p)} = e^{o(\log p)}$ uniformly in $\operatorname{Re} s \leqslant \delta$. Hence $e^{s\mathfrak{h}(p)}/p = o(1)$ and so $\log(1 + (e^{s\mathfrak{h}(p)} - 1)/p) \ll e^{\delta\mathfrak{h}(p)}/p$. Using this inequality and breaking up our sum into "dyadic" intervals, we obtain, uniformly in $\operatorname{Re} s \leqslant \delta$,

$$\begin{aligned}
\sum_{\substack{p > x \\ \mathfrak{h}(p) \neq 0}} \log\left(1 + \frac{e^{s\mathfrak{h}(p)} - 1}{p}\right) &\ll \sum_{\substack{p > x \\ \mathfrak{h}(p) \neq 0}} \frac{e^{\delta\mathfrak{h}(p)}}{p} \leqslant \sum_{k \geqslant \log x} e^{-k} \sum_{\substack{e^k \leqslant p \leqslant e^{k+1} \\ \mathfrak{h}(p) \neq 0}} e^{\delta\mathfrak{h}(p)} \\
&= \sum_{k \geqslant \log x} e^{-k} \cdot \Bigg[\sum_{\substack{e^k \leqslant p \leqslant e^{k+1} \\ 0 < \mathfrak{h}(p) \leqslant \log\log p}} e^{\delta\mathfrak{h}(p)} + \sum_{A \geqslant 1} \sum_{\substack{e^k \leqslant p \leqslant e^{k+1} \\ A \leqslant \mathfrak{h}(p)/\log\log p \leqslant A+1}} e^{\delta\mathfrak{h}(p)} \Bigg]
\end{aligned} \tag{4.41}$$

Bounding the sum over $0 < \mathfrak{h}(p) \leqslant \log\log p$ boils down to using the previous lemma. Note that under the condition $p \leqslant e^{k+1}$ and $\mathfrak{h}(p) \leqslant \log\log p$ we have $e^{\delta\mathfrak{h}(p)} \leqslant (k+1)^\delta$.

Therefore

$$\sum_{\substack{e^k \leqslant p \leqslant e^{k+1} \\ 0 < \mathfrak{h}(p) \leqslant \log\log p}} e^{\delta \mathfrak{h}(p)} \leqslant (k+1)^{\delta} \sum_{\substack{p \leqslant e^{k+1} \\ \mathfrak{h}(p) \neq 0}} 1 = O_C\left(k^{\delta} \cdot e^k \cdot k^{-C}\right)$$

where in the last inequality we used Lemma 4.18. We chose $C = 2\delta$, and conclude that the sum over $0 < \mathfrak{h}(p) \leqslant \log\log p$ in (4.41) is bounded by $e^k \cdot k^{-\delta}$.

To bound the double sum over $A \geqslant 1$ and $A \leqslant \mathfrak{h}(p)/\log\log p \leqslant A+1$ in (4.41) we will use assumption (1.3). Note that under the conditions $e^k \leqslant p \leqslant e^{k+1}$ and $A \leqslant \mathfrak{h}(p)/\log\log p \leqslant A+1$ we have $e^{\delta\mathfrak{h}(p)} \leqslant (k+1)^{\delta(A+1)}$. It follows that

$$\sum_{A \geqslant 1} \sum_{\substack{e^k \leqslant p \leqslant e^{k+1} \\ A \leqslant \mathfrak{h}(p)/\log\log p \leqslant A+1}} e^{\delta \mathfrak{h}(p)} \leqslant \sum_{A \geqslant 1} (k+1)^{\delta(A+1)} \sum_{\substack{e^k \leqslant p \leqslant e^{k+1} \\ A \log\log p \leqslant \mathfrak{h}(p)}} 1 \tag{4.42}$$

Regarding the innermost sum we proceed as follows: since $e^k \leqslant p$ we overestimate a little by replacing $A \log\log p \leqslant \mathfrak{h}(p)$ with $A \log k \leqslant \mathfrak{h}(p)$. Furthermore since $A \log k \leqslant \mathfrak{h}(p)$ implies $A \log k \leqslant f(p)$ we overestimate even more by replacing $A \log k \leqslant \mathfrak{h}(p)$ with $A \log k \leqslant f(p)$. From there, it follows that the sum in (4.42) is bounded by

$$\leqslant \sum_{A \geqslant 1} (k+1)^{\delta(A+1)} \sum_{\substack{p \leqslant e^{k+1} \\ A \log k \leqslant f(p)}} 1 \ll_B \sum_{A \geqslant 1} k^{\delta(A+1)} \cdot e^k e^{-B(A \log k)} \tag{4.43}$$

where in the last line we used the assumption (1.3). In our upper bound we choose $B = 3\delta$ and then (4.43) becomes $\ll e^k \sum_{A \geqslant 1} k^{\delta(A+1) - 3\delta A} \ll e^k \cdot k^{-\delta}$, $(A \geqslant 1)$. Collecting (4.42) and (4.43) it follows that the double sum over $A \geqslant 1$ and $A \leqslant \mathfrak{h}(p)/\log\log p \leqslant A+1$ in (4.41), is bounded by $e^k \cdot k^{-\delta}$.

Putting together our bounds, we conclude that the whole sum in (4.41) is less than $\ll \sum_{k \geqslant \log x} e^{-k} \cdot \left[e^k k^{-\delta} + e^k k^{-\delta}\right] \ll (\log x)^{-\delta+1} \ll (\log x)^{-1/2}$ since we assumed $\delta \geqslant 2$. It follows that

$$\sum_{p > x} \log\left(1 + \frac{e^{s\mathfrak{h}(p)} - 1}{p}\right) \ll (\log x)^{-1/2}$$

uniformly in $\operatorname{Re} s \leqslant \delta$ (where $\delta \geqslant 2$ without loss of generality). By the remarks made at the beginning of the lemma, the claim follows. $\square$

An important consequence of lemma 4.19 and lemma 4.18 is that $L(\mathfrak{g}; s)$ is entire.

Lemma 4.20. *Let* $f \in \mathcal{C}$. *Suppose that* $\Psi(f;t)$ *is lattice distributed on* $\mathbb{Z}$. *Then the function* $L(\mathfrak{g}; z)$ *is entire. Furthermore given* $C > 0$, *there is a* $x_0(C)$ *such that uniformly in* $|\operatorname{Re} s| \leqslant C$, $|\operatorname{Im} s| \leqslant 2\pi$ *and* $x \geqslant x_0(C)$,

$$\prod_{p \leqslant x} \left(1 - \frac{1}{p}\right)^{\hat{\Psi}(f;s)} \left(1 + \frac{e^{s\mathfrak{g}(p)}}{p-1}\right) = L(\mathfrak{g}; s) \cdot \left(1 + O_C\left((\log x)^{-1/2}\right)\right)$$

Proof. Because $\Psi(f;t)$ is lattice distributed on $\mathbb{Z}$ it has jumps on the integers. Therefore $\widehat{\Psi}(f;z) = \sum_{k\geqslant 0}\lambda_k e^{zk}$ with $\lambda_k \geqslant 0$. In particular $\widehat{\Psi}(f;\nu+it)$ is 2π-periodic in the t variable. Hence

$$L(\mathfrak{g};\nu+it) = \prod_p \left(1-\frac{1}{p}\right)^{\widehat{\Psi}(f;\nu+it)}\left(1+\frac{e^{(\nu+it)\mathfrak{g}(p)}}{p-1}\right)$$

is 2π periodic in the t variable (if the above equation is not clear recall that $\Psi(f;t) = \Psi(\mathfrak{g};t)$ by lemma 4.18, hence $\widehat{\Psi}(f;z) = \widehat{\Psi}(\mathfrak{g};z)$). Therefore, to prove that $L(\mathfrak{g};s)$ is entire, it's enough to prove that $L(\mathfrak{g};s)$ is analytic in $|\operatorname{Im} s| \leqslant 2\pi$. Given $C > 0$, consider s in the region $\mathcal{D}(C) := \{s : |\operatorname{Re} s| \leqslant C \text{ and } |\operatorname{Im} s| \leqslant 2\pi\}$. Note that

$$\begin{aligned} &\prod_{p\leqslant x}\left(1-\frac{1}{p}\right)^{\widehat{\Psi}(f;s)}\cdot\left(1+\frac{e^{sf(p)}}{p-1}\right) \\ = &\prod_{p\leqslant x}\left(1-\frac{1}{p}\right)^{\widehat{\Psi}(f;s)}\cdot\left(1+\frac{e^{s\mathfrak{g}(p)}}{p-1}\right)\prod_{p\,\leqslant\, x}\left(1-\frac{1}{p}\right)\cdot\left(1+\frac{e^{s\mathfrak{h}(p)}}{p-1}\right)\end{aligned}$$

By lemma 4.4 the first product equals $L(f;s)\cdot(1+O_C((\log x)^{-1/2})$. By lemma 4.19 the last product equals to $G(\mathfrak{h};s)\cdot(1+O_C((\log x)^{-1/2})$ (keeping the notation of lemma 4.19). Both approximations hold uniformly in $s \in \mathcal{D}(C)$ and $x \geqslant x_0(C)$ with $x_0(C)$ large enough. Dividing by $G(\mathfrak{h};s)$ on both sides we conclude that uniformly in s such that $|\operatorname{Re} s| \leqslant C, |\operatorname{Im} s| \leqslant 2\pi$ and $G(\mathfrak{h};s) \neq 0$,

$$\prod_{p\leqslant x}\left(1-\frac{1}{p}\right)^{\widehat{\Psi}(f;s)}\cdot\left(1+\frac{e^{s\mathfrak{g}(p)}}{p-1}\right) = \frac{L(f;s)}{G(\mathfrak{h};s)}\cdot\left(1+O_C((\log x)^{-1/2}\right) \tag{4.44}$$

Note that $L(f;s)/G(\mathfrak{h};s)$ is in fact analytic in $\mathcal{D}(C)$, because if $G(\mathfrak{h};s)$ vanishes then $L(f;s)$ vanishes to the same order. Thus, by continuity (4.44) extends to all of $\mathcal{D}(C)$. Since that region is bounded, the function $L(f;s)/G(\mathfrak{h};s)$, being analytic, is bounded there. Hence (4.44) guarantees that $\prod_{p\leqslant x}(1-1/p)^{\widehat{\Psi}(f;s)}(1+e^{s\mathfrak{g}(p)}(p-1)^{-1})$ converges uniformly in $|\operatorname{Re} s| \leqslant C$, $|\operatorname{Im} s| \leqslant 2\pi$. Thus $L(\mathfrak{g};s)$ is analytic in $|\operatorname{Im} s| \leqslant 2\pi$. Since $L(\mathfrak{g};\nu+it)$ is 2π periodic in the t variable, it follows that $L(\mathfrak{g};s)$ is entire. In addition, by (4.44) we must have $L(\mathfrak{g};s) = L(f;s)/G(\mathfrak{h};s)$ and the second assertion of the lemma follows from (4.44). □

A further consequence of lemma 4.19 is that $X(\mathfrak{h}) := \sum_p \mathfrak{h}(p)X_p$ has an entire moment generating function

$$\mathbb{E}\left[e^{sX(\mathfrak{h})}\right] = \prod_p\left(1+\frac{e^{s\mathfrak{h}(p)}}{p-1}\right)\cdot\left(1-\frac{1}{p}\right)$$

Thus all moments of $X(\mathfrak{h})$ are finite, and in particular the variance of $X(\mathfrak{h})$ is finite. Hence by Kolmogorov's three series theorem $X(\mathfrak{h}) = \sum_p \mathfrak{h}(p)X_p$ converges almost surely. In the next lemma we give an explicit expression for $\mathbb{P}(X(\mathfrak{h}) \geqslant t)$.

Lemma 4.21. *Let* $f \in \mathcal{C}$. *Suppose that* $\Psi(f;t)$ *is lattice distributed on* $\mathbb{Z}$. *We have*

$$\mathbb{P}\left(X(\mathfrak{h}) \leqslant t\right) = \prod_{p \in S(\mathfrak{h})} \left(1 - \frac{1}{p}\right) \sum_{\substack{n \geqslant 1 \\ p \mid n \Rightarrow p \in S(\mathfrak{h}) \\ \mathfrak{h}(n) \leqslant t}} \frac{1}{n}$$

Proof. Since X_p is a Bernoulli random variable with $\mathbb{P}\left(X_p = 1\right) = 1/p$,

$$\mathbb{E}\left[e^{sX_p}\right] = 1 - \frac{1}{p} + \frac{e^s}{p} = \left(1 - \frac{1}{p}\right) \cdot \left(1 + \frac{e^s}{p-1}\right)$$

Since in addition the X_p's are independent, and $X(\mathfrak{h}) = \sum_p \mathfrak{h}(p) X_p$,

$$\mathbb{E}\left[e^{sX(\mathfrak{h})}\right] = \prod_p \mathbb{E}\left[e^{s\mathfrak{h}(p)X_p}\right] = \prod_p \left(1 - \frac{1}{p}\right) \cdot \left(1 + \frac{e^{s\mathfrak{h}(p)}}{p-1}\right)$$

Note if $\mathfrak{h}(p) = 0$ for a prime p then the corresponding term in the above product is 1. Thus we can restrict the product to those primes p for which $\mathfrak{h}(p) \neq 0$ or equivalently to the prime $p \in S(\mathfrak{h})$. Now let

$$F(t) = \prod_{p \in S(\mathfrak{h})} \left(1 - \frac{1}{p}\right) \sum_{\substack{n \geqslant 1 \\ p \mid n \Rightarrow p \in S(\mathfrak{h}) \\ \mathfrak{h}(n) \leqslant t}} \frac{1}{n}$$

We compute the Laplace transform $\int_{\mathbb{R}} e^{st} dF(t)$ of $F(\cdot)$,

$$\begin{aligned}
\int_{\mathbb{R}} e^{st} \cdot dF(t) &= \prod_{p \in S(\mathfrak{h})} \left(1 - \frac{1}{p}\right) \int_{\mathbb{R}} e^{st} \cdot d \sum_{\substack{n \geqslant 1 \\ p \mid n \Rightarrow p \in S(\mathfrak{h}) \\ \mathfrak{h}(n) \leqslant t}} \frac{1}{n} \\
&= \prod_{p \in S(\mathfrak{h})} \left(1 - \frac{1}{p}\right) \sum_{\substack{n \geqslant 1 \\ p \mid n \Rightarrow p \in S(\mathfrak{h})}} \frac{e^{s\mathfrak{h}(n)}}{n} \\
&= \prod_{p \in S(\mathfrak{h})} \left(1 - \frac{1}{p}\right) \cdot \left(1 + \frac{e^{s\mathfrak{h}(p)}}{p-1}\right) \\
&= \mathbb{E}\left[e^{sX(\mathfrak{h})}\right] = \int_{\mathbb{R}} e^{st} d\mathbb{P}\left(X(\mathfrak{h}) \leqslant t\right)
\end{aligned}$$

By uniqueness of Laplace transforms $F(t) = \mathbb{P}\left(X(\mathfrak{h}) \leqslant t\right)$ as desired. $\square$

By the discussion preceding the above lemma, we know that

$$\sum_{p \leqslant x} \mathfrak{h}(p) X_p \longrightarrow \sum_{p} \mathfrak{h}(p) X_p$$

almost surely. Thus the convergence also holds in distribution. In the next lemma we investigate the speed of convergence in more detail.

Lemma 4.22. *Let* $f \in \mathcal{C}$. *Suppose that* $\Psi(f;t)$ *is lattice distributed on* $\mathbb{Z}$. *Let*

$$V_{\mathfrak{h}}(x;t) \;=\; \mathbb{P}\left(\sum_{p \leqslant x} \mathfrak{h}(p) X_p \geqslant t\right)$$

Then $V_{\mathfrak{h}}(x;t) = V_{\mathfrak{h}}(\infty;t) + O(V_{\mathfrak{h}}(\infty;t)^{1/4} \cdot (\log x)^{-1/2})$ *uniformly in* $t \in \mathbb{R}$.

Proof. Let $S(\mathfrak{h}) = \{p : \mathfrak{h}(p) \neq 0\}$. Proceeding as in Lemma 4.21 we find that

$$V_{\mathfrak{h}}(x;t) \;=\; \prod_{\substack{p \in S(\mathfrak{h}) \\ p \leqslant x}} \left(1 - \frac{1}{p}\right) \sum_{\substack{n \geqslant 1 \\ p | n \Rightarrow p \in S(\mathfrak{h}),\, p \leqslant x \\ \mathfrak{h}(n) \geqslant t}} \frac{1}{n} \tag{4.45}$$

We first complete the product over $p \leqslant x$ to a product over all primes in $S(\mathfrak{h})$. First of all $\prod_{p > x, p \in S} (1 - 1/p)^{-1} \geqslant 1$. Upon expanding the Euler product we find

$$1 \leqslant \prod_{\substack{p \in S(\mathfrak{h}) \\ p > x}} \left(1 - \frac{1}{p}\right)^{-1} \leqslant 1 + \sum_{\substack{n \geqslant x \\ p | n \Rightarrow p \in S(\mathfrak{h})}} \frac{1}{n} \leqslant 1 + \frac{1}{\log x} \sum_{\substack{n \geqslant 1 \\ p | n \Rightarrow p \in S(\mathfrak{h})}} \frac{\log n}{n}$$

The rightmost sum converges by the corollary to lemma 4.18. Thus $\prod_{p > x, p \in S(\mathfrak{h})} (1 - 1/p)$ equals to $1 + O(1/\log x)$. Hence (4.45) becomes

$$V_{\mathfrak{h}}(x;t) = \prod_{p \in S} \left(1 - \frac{1}{p}\right) \sum_{\substack{n \geqslant 1 \\ p | n \Rightarrow p \in S,\, p \leqslant x \\ \mathfrak{h}(n) \geqslant t}} \frac{1}{n} + O\left(\frac{V_f(x;t)}{\log x}\right)$$

Forgetting about $p \leqslant x$ in the above formula, we obtain, $V_{\mathfrak{h}}(x;t) \leqslant V_{\mathfrak{h}}(\infty;t) + O(V_{\mathfrak{h}}(x;t)/\log x)$. Iterating this inequality gives $V_{\mathfrak{h}}(x;t) \leqslant V_{\mathfrak{h}}(\infty;t) + O(V_{\mathfrak{h}}(\infty;t)/\log x)$. To obtain a lower bound for $V_{\mathfrak{h}}(x;t)$ we bound $V_{\mathfrak{h}}(\infty;t) - V_{\mathfrak{h}}(x;t)$ from above. By the previous equation

$$V_{\mathfrak{h}}(\infty;t) - V_{\mathfrak{h}}(x;t) \;=\; \prod_{p \in S} \left(1 - \frac{1}{p}\right) \sum_{\substack{p | n \Rightarrow p \in S(\mathfrak{h}) \\ \exists p | n : p > x \\ \mathfrak{h}(n) \geqslant t}} \frac{1}{n} + O\left(\frac{V_f(x;t)}{\log x}\right) \tag{4.46}$$

We overestimate the above sum by replacing the condition $\exists p | n : p > x$ with $n > x$. Then we apply Cauchy-Schwarz, singling out $n > x$ in one term and the remaining condition in the second term. We select weights so as to make the sum over $n > x$ convergent. In more detail, we bound (4.46) by

$$\left(\sum_{n>x} \frac{1}{n\,(\log n)^2}\right)^{1/2} \cdot \left(\sum_{\substack{m \geqslant 1 \\ p|m \Rightarrow p \in S(\mathfrak{h}) \\ \mathfrak{h}(m) \geqslant t}} \frac{(\log m)^2}{m}\right)^{1/2} \tag{4.47}$$

The sum on the left is $\ll 1/(\log x)$. To bound the sum on the right we apply once again Cauchy-Schwarz, obtaining the following bound

$$\left(\sum_{\substack{m \geqslant 1 \\ p|m \Rightarrow p \in S(\mathfrak{h}) \\ \mathfrak{h}(m) \geqslant t}} \frac{1}{m}\right)^{1/2} \cdot \left(\sum_{\substack{m \geqslant 1 \\ p|m \Rightarrow p \in S(\mathfrak{h})}} \frac{(\log m)^4}{m}\right)^{1/2} \tag{4.48}$$

By the corollary to lemma 4.18 the sum over $m \geqslant 1$ is $O(1)$. By lemma 4.21 the sum on the left is $C\mathbb{P}(X(\mathfrak{h}) \geqslant t)$ for some constant $C > 0$. Thus the above is bounded by $\ll \mathbb{P}(X(\mathfrak{h}) \geqslant t)^{1/2}$. By (4.46), (4.47) and (4.48), $V_{\mathfrak{h}}(\infty;t) - V_{\mathfrak{h}}(x;t) \leqslant O((\log x)^{-1/2} V_{\mathfrak{h}}(\infty;t)^{1/4})$. On the other hand $0 \leqslant V_{\mathfrak{h}}(x;t) \leqslant V_{\mathfrak{h}}(\infty;t) + O(V_{\mathfrak{h}}(\infty;t)/\log x)$. The lemma follows. $\square$

The next result is a rather technical corollary to the above lemma. It shows that we can modify the random variable $\sum_{p\leqslant x} \mathfrak{h}(p)X_p$ on the primes $p > \xi(x)$ ($\xi(x) \to \infty$) without destroying uniform convergence (in distribution) to $\sum_p \mathfrak{h}(p)X_p$.

Corollary 4.23. *Let $f \in \mathcal{C}$. Suppose that $\Psi(f;t)$ is lattice distributed on $\mathbb{Z}$. Let $y \to \infty$ as $x \to \infty$ but with $y \leqslant x$. Let $\mathfrak{H}$ be a strongly additive function defined by*

$$\mathfrak{H}(p) = \begin{cases} \lceil \mathfrak{h}(p) \rceil & \textit{if } p \geqslant y \\ \mathfrak{h}(p) & \textit{otherwise} \end{cases}$$

We have, uniformly in $t \in \mathbb{R}$,

$$\mathbb{P}\left(\sum_{p\leqslant x} \mathfrak{H}(p)X_p \geqslant t\right) = \mathbb{P}\left(\sum_{p} \mathfrak{h}(p)X_p \geqslant t\right) + O_A\left(\frac{e^{-At}}{(\log y)^{1/4}}\right)$$

for any given $A > 0$.

Proof. We retain the notation $V_{\mathfrak{h}}(x;t)$ from the previous lemma. Since $\mathfrak{H} \geqslant \mathfrak{h}$ and $x \geqslant y$,

$$\mathbb{P}\left(\sum_{p\leqslant x} \mathfrak{H}(p)X_p \geqslant t\right) \geqslant \mathbb{P}\left(\sum_{p\leqslant y} \mathfrak{h}(p)X_p \geqslant t\right) = V_{\mathfrak{h}}(\infty;t) + O\left(\frac{V_{\mathfrak{h}}(\infty;t)^{1/2}}{(\log y)^{1/4}}\right) \tag{4.49}$$

where in the second equality we used the previous lemma. Further by lemma 4.19 $X(\mathfrak{h}) := \sum_p \mathfrak{h}(p)X_p$ has an entire moment generating function. Hence, by Chernoff's bound $V_{\mathfrak{h}}(\infty;t) \leqslant \mathbb{E}[e^{AX(\mathfrak{h})}]e^{-At} = O_A(e^{-At})$ for any given $A > 0$. We conclude that the error term in (4.49) is bounded by $O_A(e^{-At} \cdot (\log y)^{-1/4})$. To derive the upper bound let us note that lemma 4.21

and lemma 4.22 also holds for the additive function $\mathfrak{H}$ (the important observation here is that $\mathfrak{H}(p)$ vanishes exactly when $\mathfrak{h}(p)$ so $\mathfrak{H}$ is a "small" additive function). Therefore

$$\mathbb{P}\left(\sum_{p\leqslant x}\mathfrak{H}(p)X_p \geqslant t\right) \quad\leqslant\quad V_{\mathfrak{H}}(\infty;t) = V_{\mathfrak{H}}(y;t) + O\left(\frac{V_{\mathfrak{H}}(\infty;t)^{1/2}}{(\log y)^{1/4}}\right) \tag{4.50}$$

Now $X(\mathfrak{H}) := \sum_p \mathfrak{H}(p)X_p$ also has an entire moment generating function. Hence by Chernoff's bound $V_{\mathfrak{H}}(\infty;t) = O_A(e^{-At})$ for any given $A > 0$. It follows that the error term in (4.50) is $O_A(e^{-At}\cdot(\log y)^{-1/4})$. By definition of $\mathfrak{H}$ we have the equality $V_{\mathfrak{H}}(y;t) = V_{\mathfrak{h}}(y;t)$. By lemma 4.22 and the bound $V_{\mathfrak{h}}(\infty;t) \ll_A e^{-At}$,

$$V_{\mathfrak{h}}(y;t) = V_{\mathfrak{h}}(\infty;t) + O_A\left(e^{-At}\cdot(\log y)^{-1/4}\right)$$

It follows that $V_{\mathfrak{H}}(y;t) = V_{\mathfrak{h}}(\infty;t) + O_A(e^{-At}\cdot(\log y)^{-1/4})$. On combining this equality with (4.50) we obtain the desired upper bound

$$\mathbb{P}\left(\sum_{p\leqslant x}\mathfrak{H}(p)X_p \geqslant t\right) \leqslant V_{\mathfrak{h}}(\infty;t) + O_A\left(\frac{e^{-At}}{(\log y)^{1/4}}\right)$$

We also established a lower bound of the same quality, hence the lemma follows. ☐

4.5.2. *Computing a "saddle-point integral".* The goal of this section is to prove the following lemma.

Lemma 4.24. *Let $f \in \mathcal{C}$. Let $\mathcal{A}(s)$ be analytic in $\operatorname{Re} s \geqslant 0$, and suppose that $\mathcal{A}(x)$ does not vanish for $x \geqslant 0$. Suppose that $\Psi(f;t)$ is lattice distributed on $\mathbb{Z}$. Let $s := \nu + it$ with both ν, t real and $\nu := \nu_f(x;\Delta)$. Given $\delta, \varepsilon > 0$, we have*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathcal{A}(s)\cdot(1/s)\mathcal{P}_{\mathfrak{h}}(\xi_f(x;\Delta);s)\cdot(\log x)^{\widehat{\Psi}(f;s)-1}\cdot e^{-s\xi_f(x;\Delta)}\cdot dt \tag{4.51}$$

$$= \quad \mathcal{A}(\nu)(1/\nu)\mathcal{P}_{\mathfrak{h}}(\xi_f(x;\Delta);\nu)\cdot\frac{(\log x)^{\widehat{\Psi}(f;\nu)-1-\nu\widehat{\Psi}'(f;\nu)}}{(2\pi\widehat{\Psi}''(f;\nu)\log\log x)^{1/2}}\cdot e^{-\nu c(f)}\cdot(1+o(1))$$

uniformly for Δ in the range $(\log\log x)^{\varepsilon} \ll \Delta \leqslant \delta\sigma_{\Psi}(f;x)$.

First we need to show that $\mathcal{P}_{\mathfrak{h}}(a;s)$ behaves "as an analytic function".

Lemma 4.25. *Let $f \in \mathcal{C}$. Suppose that $\Psi(f;t)$ is lattice distributed on $\mathbb{Z}$. Given $C > 0$, uniformly in $|\delta| \leqslant \pi$, $0 \leqslant \nu \leqslant C$ and $0 \leqslant a \leqslant 1$,*

$$\mathcal{P}_{\mathfrak{h}}(a;\nu+\delta) = \mathcal{P}_{\mathfrak{h}}(a;\nu) + O_C(\delta)$$

Here δ is allowed to be a complex number. Furthermore $\nu/(e^{\nu}-1) \leqslant \mathcal{P}_{\mathfrak{h}}(a;\nu) = O_C(1)$ uniformly in $0 \leqslant a \leqslant 1$ and $0 \leqslant \nu \leqslant C$.

Remark. The restriction $0 \leqslant a \leqslant 1$ is unnecessary because $\mathcal{P}_{\mathfrak{h}}(a;\nu)$ is 1-periodic in the a variable.

Proof. As usual write $X(\mathfrak{h}) := \sum_p \mathfrak{h}(p)X_p$. By definition we have

$$\mathcal{P}_{\mathfrak{h}}(a; \nu+\delta) \;=\; (\nu+\delta)\sum_{\ell\in\mathbb{Z}} e^{(\nu+\delta)\cdot(\ell+a)} \cdot \mathbb{P}\left(X(\mathfrak{h}) \geqslant \ell + a\right)$$

We split the sum at $\ell < 0$ and $\ell \geqslant 0$ and handle separately the two ranges. When $\ell < 0$ we have $\mathbb{P}(X(\mathfrak{h}) \geqslant \ell + a) = 1$. Note also that $z/(e^z - 1)$ is analytic in the region $\{\nu + it : \nu \in \mathbb{R}, |t| \leqslant \pi\}$. Therefore $(\nu+\delta)/(e^{(\nu+\delta)} - 1) = \nu/(e^\nu - 1) + O(\delta)$ (the implicit constant depends on C, we won't bother making that dependence explicit). With those two remarks in mind, the sum over $\ell < 0$ contributes

$$\begin{aligned}
(\nu+\delta)\sum_{\ell>0} e^{(-\ell+a)\cdot(\nu+\delta)} &= \frac{\nu+\delta}{e^{\nu+\delta}-1}\cdot e^{a\cdot(\nu+\delta)} \\
&= \left(\frac{\nu}{e^\nu - 1} + O(\delta)\right)\cdot e^{a\nu}\cdot(1+O(\delta)) \\
&= \frac{\nu\cdot e^{a\nu}}{e^\nu - 1} + O(\delta) \;=\; \nu\sum_{\ell>0} e^{(-\ell+a)\cdot\nu} + O(\delta) \qquad (4.52)
\end{aligned}$$

We split the sum over $\ell \geqslant 0$,

$$(\nu+\delta)\sum_{\ell\geqslant 0} e^{(\nu+\delta)\cdot(\ell+a)}\cdot\mathbb{P}\left(X(\mathfrak{h}) \geqslant \ell + a\right) \qquad (4.53)$$

into $0 \leqslant \ell < |\delta|^{-1}$ and $\ell > |\delta|^{-1}$. When $0 \leqslant \ell \leqslant |\delta|^{-1}$ we have

$$\begin{aligned}
(\nu+\delta)e^{(\nu+\delta)(\ell+a)} &= \nu e^{(\nu+\delta)(\ell+a)} + O\left(\delta e^{(\nu+\pi)(\ell+1)}\right) \\
&= \nu e^{\nu(\ell+a)}\cdot(1+O(\delta\ell)) + O\left(\delta e^{(\nu+\pi)(\ell+1)}\right) \\
&= \nu e^{\nu(\ell+a)} + O\left(\delta\ell e^{\nu(\ell+1)} + \delta e^{(\nu+\pi)(\ell+1)}\right) \\
&= \nu e^{\nu(\ell+a)} + O\left(\delta\ell e^{(\nu+\pi)(\ell+1)}\right) \qquad (4.54)
\end{aligned}$$

Splitting the sum (4.53) into $0 \leqslant \ell < |\delta|^{-1}$ and $\ell > |\delta|^{-1}$, and using (4.54), we obtain that (4.53) equals to

$$\begin{aligned}
&(\nu+\delta)\sum_{0\leqslant\ell\leqslant|\delta|^{-1}} e^{(\nu+\delta)(\ell+a)}\cdot\mathbb{P}(X(\mathfrak{h})\geqslant \ell+a) + O\left(\sum_{\ell\geqslant|\delta|^{-1}} e^{(\nu+\pi)(\ell+1)}\mathbb{P}(X(\mathfrak{h})\geqslant\ell)\right) \\
&= \nu\sum_{0\leqslant\ell\leqslant|\delta|^{-1}} e^{\nu(\ell+a)}\cdot\mathbb{P}(X(\mathfrak{h})\geqslant\ell+a) + O\left(\sum_{\ell\geqslant 0}\delta\ell e^{(\nu+\pi)(\ell+1)}\mathbb{P}(X(\mathfrak{h})\geqslant\ell)\right) \\
&= \nu\sum_{\ell\geqslant 0} e^{\nu(\ell+a)}\cdot\mathbb{P}(X(\mathfrak{h})\geqslant\ell+a) + O\left(\sum_{\ell\geqslant 0}\delta\ell e^{(\nu+\pi)(\ell+1)}\mathbb{P}(X(\mathfrak{h})\geqslant\ell)\right) \qquad (4.55)
\end{aligned}$$

By lemma 4.19 $X(\mathfrak{h})$ has an entire moment generating function. Therefore for each fixed $A > 0$, we have $\mathbb{P}(X(\mathfrak{h}) \geqslant t) \leqslant \mathbb{E}[e^{AX(\mathfrak{h})}]e^{-At} = O_A(e^{-At})$. In particular we have $\mathbb{P}(X(\mathfrak{h}) \geqslant$

$\ell) = O_C(e^{-(C+2+\pi)(\ell+1)}) = O_C(e^{-(\nu+2+\pi)(\ell+1)})$. Thus the error term in (4.55) is $O_C(\delta)$. Adding up the estimate (4.55) and (4.52), the first assertion of the lemma follows.

The lower bound in the second assertion follows from

$$\mathcal{P}_{\mathfrak{h}}(a;\nu) \quad \geqslant \quad \nu \sum_{\ell \leqslant 0} e^{\nu(\ell+a)} \; = \; \frac{\nu e^{a\nu}}{e^{\nu}-1} \; \geqslant \; \frac{\nu}{e^{\nu}-1}$$

For the upper bound, recall that $\mathbb{P}(X(\mathfrak{h}) \geqslant \ell) = O_C(e^{-(C+1)(\ell+1)})$. Therefore

$$\mathcal{P}_{\mathfrak{h}}(a;\nu) \quad \leqslant \quad \nu \sum_{\ell < 0} e^{\nu(\ell+a)} + \nu \sum_{\ell \geqslant 0} e^{C\cdot(\ell+1)} \cdot \mathbb{P}(X(\mathfrak{h}) \geqslant \ell) \ll_C \frac{\nu}{e^{\nu}-1} + \nu = O_C(1)$$

The lemma is now proven. $\square$

Lemma 4.26. *Let* $f \in \mathcal{C}$. *Suppose that* $\Psi(f;t)$ *is lattice distributed on* $\mathbb{Z}$. *Given* $\varepsilon > 0$ *there is a* $\delta > 0$ *such that*

$$|\exp(\widehat{\Psi}(f;\nu+it) - \widehat{\Psi}(f;\nu))| \leqslant 1-\delta$$

for all $\pi \geqslant |t| \geqslant \varepsilon$ *and* $\nu \geqslant 0$.

Proof. For any $\nu, t \in \mathbb{R}$ we have

$$\begin{aligned} \mathrm{Re}\left(\widehat{\Psi}(f;\nu+it) - \widehat{\Psi}(f;\nu)\right) &= \mathrm{Re}\left(\left(\int_{\mathbb{R}} e^{\nu u} \cdot \left(e^{itu}-1\right) d\Psi(f;u)\right)\right) \\ &= \int_{\mathbb{R}} e^{\nu u} \cdot (\cos(tu)-1)\, d\Psi(f;u) \\ &\leqslant \int_{\mathbb{R}} (\cos(tu)-1)\, d\Psi(f;u) \;=\; \mathrm{Re}\left(\widehat{\Psi}(f;it)-1\right) \quad (4.56) \end{aligned}$$

Therefore $|\exp(\widehat{\Psi}(f;\nu+it) - \widehat{\Psi}(f;it))| \leqslant |\exp(\widehat{\Psi}(f;it)-1)|$ and it's enough to show that given $\varepsilon > 0$ there is a $\delta > 0$ such that $|\exp(\widehat{\Psi}(f;it)-1)| \leqslant 1-\delta$ for all $\pi \geqslant |t| \geqslant \varepsilon$. Since $\Psi(f;t)$ is lattice distributed it has jumps at the integers $0,1,2,\ldots$ (there are no jumps at the negative integers because $f \geqslant 0$). Denote the size of each jump by $\lambda_0, \lambda_1, \ldots$. Thus

$$\widehat{\Psi}(f;z) = \int_{\mathbb{R}} e^{zt} d\Psi(f;t) = \sum_{k \geqslant 0} \lambda_k \cdot e^{zk}$$

If $\Psi(f;t)$ has all its mass concentrated at one integer k, then $k=1$ and $\widehat{\Psi}(f;z) = e^z$. In this case the bound $|\exp(\widehat{\Psi}(f;it)-1)| = \exp(\cos(t)-1) \leqslant 1-\delta$ for $\pi \geqslant |t| \geqslant \varepsilon$ is trivial. In the remaining case there are at least two k,ℓ for which $\lambda_k > 0$ and $\lambda_\ell > 0$. Without loss of generality we can assume that $(k,\ell) = 1$. Otherwise all the k for which $\lambda_k > 0$ would be divisible by a common prime p; thus $\Psi(f;t)$ would not be lattice distributed on $\mathbb{Z}$ (but on $p\mathbb{Z}$). Thus for two such k,ℓ, we have

$$\mathrm{Re}(\widehat{\Psi}(f;it)-1) = \sum_{r \geqslant 0} \lambda_r \cdot (\cos(rt)-1) \leqslant \lambda_k \cdot (\cos(kt)-1) + \lambda_\ell \cdot (\cos(\ell t)-1) \leqslant 0$$

We claim that for $|t| \leqslant \pi$ the upper bound is attained only at $t = 0$. Indeed suppose that $\lambda_k(\cos(kt) - 1) + \lambda_\ell(\cos(\ell t) - 1) = 0$. Then simultaneously $\cos(kt) = 1$ and $\cos(\ell t) = 1$. Hence $t = 2\pi\alpha/\ell$ and $t = 2\pi\beta/k$ for some integer $|\alpha| \leqslant \ell/2$ and some integer $|\beta| \leqslant k/2$, because $|t| \leqslant \pi$. In particular $2\pi\alpha/\ell = 2\pi\beta/k$. If $t \neq 0$ then $\alpha \neq 0$ and $\beta \neq 0$, hence, $k\alpha/\ell = \beta \in \mathbb{Z}$ which is impossible because $(k, \ell) = 1$ and $|\alpha| < \ell$. Thus $\mathrm{Re}(\widehat{\Psi}(f; it) - 1) < 0$ for all $0 < |t| \leqslant \pi$. Hence we have $|\exp(\widehat{\Psi}(f; it) - 1)| < 1$ for all $0 < |t| \leqslant \pi$. By continuity, given $\varepsilon > 0$, there is a $\delta > 0$ such that $|\exp(\widehat{\Psi}(f; it) - 1)| \leqslant 1 - \delta$ for all $\varepsilon \leqslant |t| \leqslant \pi$. Since

$$|\exp(\widehat{\Psi}(f; \nu + it) - \widehat{\Psi}(f; \nu)| \leqslant |\exp(\widehat{\Psi}(f; it) - 1)|$$

for all $\nu \in \mathbb{R}$, the lemma follows. $\square$

We are ready to prove the "general lemma".

Proof of Lemma 4.24. The function $\omega(f; t)$ is continuous, hence the parameter $\nu := \nu_f(x; \Delta) = \omega(f; \Delta/\sigma_\Psi(f; x))$ is bounded throughout $1 \leqslant \Delta \leqslant \delta\sigma_\Psi(f; x)$, the bound depending only on δ. To evaluate (4.51) we proceed with the saddle-point method. As is usually done we split the integral into two ranges. The "tiny" range $|t| \leqslant \lambda(x)(\log\log x)^{-1/2}$ denoted by $\mathcal{M}$, because this range will contribute the main term, and the remaining range $|t| \geqslant \lambda(x)(\log\log x)^{-1/2}$ denoted $\mathcal{R}$. Here we choose $\lambda(x)$ to be any function such that $(\log\log\log x)^4 \ll \lambda(x) \ll (\log\log\log x)^5$. Let us confine attention to how the integrand in (4.51) behaves when $t \in \mathcal{M}$. First of all, when $t \in \mathcal{M}$, upon expanding $\widehat{\Psi}(f; \nu + it)$ into a Taylor series we obtain

$$\widehat{\Psi}(f; \nu + it) \;=\; \widehat{\Psi}(f; \nu) + it\,\widehat{\Psi}'(f; \nu) - \frac{t^2}{2} \cdot \widehat{\Psi}''(f; \nu) + O_\delta\left(\lambda(x)^3 (\log_2 x)^{-3/2}\right)$$

The δ in the error term comes from the bound $0 \leqslant \nu = O_\delta(1)$ on ν. We will not indicate the dependence on δ in our error terms. Using the above expansion and Lemma 4.8 we conclude that

$$\begin{aligned}
& (\log x)^{\widehat{\Psi}(f;\nu+it)} \cdot e^{-it\,\xi_f(x;\Delta)} \\
= \;& (\log x)^{\widehat{\Psi}(f;\nu)+it\widehat{\Psi}'(f;\nu)-t^2/2\cdot\widehat{\Psi}''(f;\nu)}\, e^{-it\xi_f(x;\Delta)} \left(1 + O(\lambda(x)^3(\log_2 x)^{-1/2})\right) \\
= \;& (\log x)^{\widehat{\Psi}(f;\nu)+it\widehat{\Psi}'(f;\nu)-t^2/2\cdot\widehat{\Psi}''(f;\nu)}\, (\log x)^{-it\widehat{\Psi}'(f;\nu)}\, e^{-itc(f)} \left(1 + O(\lambda(x)^3(\log_2 x)^{-1/2}\right) \\
= \;& (\log x)^{\widehat{\Psi}(f;\nu)-(t^2/2)\cdot\widehat{\Psi}''(f;\nu)}\, e^{-itc(f)} \left(1 + O\left(\lambda(x)^3(\log_2 x)^{-1/2}\right)\right) \\
= \;& (\log x)^{\widehat{\Psi}(f;\nu)-(t^2/2)\cdot\widehat{\Psi}''(f;\nu)} \cdot \left(1 + O\left(\lambda(x)^3(\log_2 x)^{-1/2}\right)\right) \qquad (4.57)
\end{aligned}$$

where in the last line we used the expansion $e^{itc(f)} = 1 + O(t \cdot c(f))$ together with the bound $|t| \leqslant \lambda(x)(\log_2 x)^{-1/2}$. Note that $|t/\nu| \leqslant |\lambda(x) \cdot (\log\log x)^{-1/2} \cdot \nu^{-1}| = o(1)$ because

$\nu \asymp \Delta \cdot (\log\log x)^{-1/2}$ and $\lambda(x) = o(\Delta)$. By lemma 4.25, when $t \in \mathcal{M}$,

$$\begin{aligned}
& (1/(\nu + it)) \cdot \mathcal{P}_{\mathfrak{h}}\left(\xi_f(x;\Delta); \nu + it\right) \\
= \; & \nu^{-1} \cdot (1 + O(t \cdot \nu^{-1})) \cdot (\mathcal{P}_{\mathfrak{h}}(\xi_f(x;\Delta);\nu) + O(t)) \\
= \; & \nu^{-1} \cdot \mathcal{P}_{\mathfrak{h}}(\xi_f(x;\Delta);\nu) \cdot (1 + O_\delta(t)) \cdot (1 + O(t \cdot \nu^{-1})) \\
= \; & \nu^{-1} \cdot \mathcal{P}_{\mathfrak{h}}(\xi_f(x;\Delta);\nu) \cdot (1 + O(\lambda(x) \cdot (\log\log x)^{-1/2} \cdot \nu^{-1}) \qquad (4.58)
\end{aligned}$$

The third line is justified by the bound $\mathcal{P}_{\mathfrak{h}}(a;\nu) \gg_\delta 1$ which follows from the inequality $\mathcal{P}_{\mathfrak{h}}(a;\nu) \geqslant \nu/(e^\nu - 1)$ of lemma 4.25 and $\nu = O_\delta(1)$. Finally by analyticity of $\mathcal{A}(z)$, for $t \in \mathcal{M}$, we have

$$\begin{aligned}
\mathcal{A}(\nu + it) & = \mathcal{A}(\nu) + O(t) \\
& = \mathcal{A}(\nu) + O\left(\lambda(x) \cdot (\log\log x)^{-1/2}\right) \\
& = \mathcal{A}(\nu) \cdot (1 + O\left(\lambda(x) \cdot (\log\log x)^{-1/2}\right)) \qquad (4.59)
\end{aligned}$$

where the last line is justified by the non-vanishing of $\mathcal{A}(\nu)$ (because $\mathcal{A}(x) \neq 0$ for $x \geqslant 0$ and $0 \leqslant \nu \leqslant O_\delta(1)$ we have $\mathcal{A}(\nu) \gg_\delta 1$ by continuity of $\mathcal{A}(\cdot)$). From the equations (4.57), (4.58), (4.59) and lemma 4.8, we conclude that

$$\begin{aligned}
& \mathcal{A}(\nu + it) \cdot (1/(\nu + it))\mathcal{P}_{\mathfrak{h}}(\xi_f(x;\Delta);\nu + it) \cdot (\log x)^{\widehat{\Psi}(f;\nu+it)-1} \cdot e^{-(\nu+it)\xi_f(x;\Delta)} \\
= \; & \mathcal{A}(\nu) \cdot (1/\nu)\mathcal{P}_{\mathfrak{h}}(\xi_f(x;\Delta);\nu) \cdot (\log x)^{\widehat{\Psi}(f;\nu)-1-(t^2/2)\widehat{\Psi}''(f;\nu)} \cdot e^{-\nu\xi_f(x;\Delta)} \cdot (1 + O(\mathcal{E})) \\
= \; & \mathcal{A}(\nu) \cdot (1/\nu)\mathcal{P}_{\mathfrak{h}}(\xi_f(x;\Delta);\nu) \cdot (\log x)^{A(f;\nu)-(t^2/2)\widehat{\Psi}''(f;\nu)} \cdot e^{-\nu c(f)} \cdot (1 + O\left(\mathcal{E}\right)) \qquad (4.60)
\end{aligned}$$

where $A(f;\nu) = \widehat{\Psi}(f;\nu) - 1 - \nu\widehat{\Psi}'(f;\nu)$ and $\mathcal{E} := \mathcal{E}\left(x;\nu\right) := \lambda(x)^3(\log\log x)^{-1/2} \cdot \nu^{-1}$. In view of the above relation to estimate (4.51) over $t \in \mathcal{M}$, it remains to note that

$$\begin{aligned}
& \int_{\mathcal{M}} (\log x)^{-(t^2/2)\cdot\widehat{\Psi}''(f;\nu)} \cdot \frac{dt}{2\pi} \\
= \; & \int_{\mathcal{M}} \exp\left(-\frac{t^2}{2} \cdot \widehat{\Psi}''(f;\nu)\log\log x\right) \cdot \frac{dt}{2\pi} \\
= \; & \frac{1}{(\widehat{\Psi}''(f;\nu)\log\log x)^{1/2}} \int_{-\lambda(x)}^{\lambda(x)} e^{-t^2/2} \cdot \frac{dt}{2\pi} \\
= \; & \frac{1}{(\widehat{\Psi}''(f;\nu)\log\log x)^{1/2}} \cdot \left(\int_{\mathbb{R}} e^{-t^2/2} \cdot \frac{dt}{2\pi} + o(1)\right) \\
= \; & \frac{1}{(2\pi\widehat{\Psi}''(f;\nu)\log\log x)^{1/2}} \cdot (1 + o(1)) \qquad (4.61)
\end{aligned}$$

Together from (4.60) and (4.61) we conclude that the integral (4.51) restricted to $t \in \mathcal{M}$ is equal to

$$\mathcal{A}(\nu)(1/\nu)\mathcal{P}_{\mathfrak{h}}(\xi_f(x;\Delta);\nu) \cdot \frac{(\log x)^{\widehat{\Psi}(f;\nu)-1-\nu\widehat{\Psi}'(f;\nu)}}{(2\pi\widehat{\Psi}''(f;\nu)\log\log x)^{1/2}} \cdot e^{-\nu c(f)} \cdot (1 + o(1) + O\left(\mathcal{E}(x;\nu)\right))$$

where $\mathcal{E}(x;\nu) := \lambda(x)^3(\log\log x)^{-1/2}\nu^{-1}$. As we noticed $\lambda(x)^3(\log\log x)^{-1/2}\nu^{-1} = o(1)$ because $\lambda(x)^3 = o(\Delta)$ and $\nu \asymp \Delta/(\log\log x)^{1/2}$. Therefore the above formula furnishes the desired main term. It remains to bound the integral (4.51) restricted to $t \in \mathcal{R}$. By lemma 4.25 we have $\mathcal{P}_{\mathfrak{h}}(a;\nu+it) = \mathcal{P}_{\mathfrak{h}}(a;\nu) + O(|t|) \ll 1$ uniformly in $1 \leqslant \Delta \leqslant c\sigma(f;x)$ because $\nu \asymp \Delta/\sigma(f;x)$ belongs to a bounded range. Further since $\mathcal{A}(\cdot)$ is analytic we have $\mathcal{A}(\nu+it) \ll 1$ because $\nu+it$ lies in a bounded domain. Thus (writing $s = \nu+it$)

$$\begin{aligned} &\int_{\mathcal{R}} \mathcal{A}(s)(1/s)\mathcal{P}_{\mathfrak{h}}(\xi_f(x;\Delta);s)\cdot(\log x)^{\hat{\Psi}(f;s)-1}\cdot e^{-s\xi_f(x;\Delta)}\cdot dt \\ \ll\quad & (1/\nu)\cdot(\log x)^{\Psi(f;\nu)-1}\cdot e^{-\nu\xi_f(x;\Delta)}\cdot\int_{\mathcal{R}}(\log x)^{\mathrm{Re}(\hat{\Psi}(f;\nu+it)-\hat{\Psi}(f;\nu))}\cdot dt \end{aligned} \tag{4.62}$$

Given $\varepsilon > 0$, by lemma 4.26 there is a $\delta > 0$ such that $|\exp(\hat{\Psi}(f;\nu+it)-\hat{\Psi}(f;\nu))| \leqslant 1-\delta$ for all $\pi \geqslant |t| \geqslant \varepsilon$. Exponentiating we get $(\log x)^{\mathrm{Re}(\hat{\Psi}(f;\nu+it)-\hat{\Psi}(f;\nu)} \leqslant (\log x)^{\log(1-\delta)}$ which is sufficient to bound the part $|\pi| \geqslant |t| \geqslant \varepsilon$ of the integral in (4.62). We are left with bounding the remaining range $\lambda(x)\cdot(\log\log x)^{-1/2} \leqslant |t| \leqslant \varepsilon$. As in the proof of Lemma 4.26, we observe that since $\Psi(f;t)$ is lattice distributed, we have

$$\hat{\Psi}(f;z) \quad = \quad \sum_{k\geqslant 0}\lambda_k\cdot e^{zk}$$

Hence $\mathrm{Re}(\hat{\Psi}(f;\nu+it)-\hat{\Psi}(f;\nu)) = \sum_{k\geqslant 0}\lambda_k\cdot e^{\nu k}\cdot(\cos(kt)-1) \leqslant \lambda_\ell(\cos(\ell t)-1)$ where $\ell > 0$ is the first integer for which $\lambda_\ell > 0$. In the range $|t| \leqslant \varepsilon$ we have $\cos(\ell t)-1 \leqslant -ct^2$ provided that ε is sufficiently small (of course c depends on ε and ℓ). We reduce ε if necessary and fix it, once it's small enough (note that our bound over $\pi \geqslant |t| \geqslant \varepsilon$ is negligible as long as ε is fixed). Thus $\mathrm{Re}(\hat{\Psi}(f;\nu+it)-\hat{\Psi}(f;\nu)) \leqslant -ct^2$ for $|t| \leqslant \varepsilon$. Hence

$$\begin{aligned} &\int_{\mathcal{R}\cap\{|t|\leqslant\varepsilon\}}(\log x)^{\mathrm{Re}\left(\hat{\Psi}(f;\nu+it)-\hat{\Psi}(f;\nu)\right)}\,dt \\ \leqslant\quad & 2\int_{\lambda(x)\cdot(\log\log x)^{-1/2}}^{\varepsilon}\exp\left(-ct^2\cdot\log\log x\right)dt \leqslant 2\cdot e^{-c\lambda(x)^2} \end{aligned}$$

Thus by (4.62) and our earlier remarks, the integral in (4.51) restricted to $t \in \mathcal{R}$, turns out to be bounded by

$$(1/\nu)\cdot(\log x)^{\hat{\Psi}(f;\nu)-1}\cdot e^{-\nu\xi_f(x;\Delta)}\cdot\left[\exp(-c\cdot\lambda(x)^2)+(\log x)^{\log(1-\delta)}\right]$$

which is negligible because $\lambda(x) \gg \log\log\log x$ and $\xi_f(x;\Delta) = \hat{\Psi}'(f;\nu)\log\log x + O(1)$ by lemma 4.8. Hence the integral (4.51) restricted to $\mathcal{R}$ is negligible. This, together with the asymptotic for "(4.51) restricted to $\mathcal{M}$" finishes the proof of the lemma. □

4.5.3. *Proof of Proposition 4.17.* We are now ready to prove Proposition 4.17.

Proof of Proposition 4.17. Let $y := y(x) \leqslant (1/8)\log\log\log x$ be a parameter growing to infinity so slow so as to have $1-(\log\log x)^{-1/8} \geqslant |\{\mathfrak{h}(p)\}-\{\mathfrak{h}(q)\}| \geqslant (\log\log x)^{-1/8}$ for any two prime $p, q \leqslant y$ with $\{\mathfrak{h}(p)\} \neq \{\mathfrak{h}(q)\}$. (Here $\{\mathfrak{h}(p)\}$ denotes the fractional part of $\mathfrak{h}(p)$). Let $\mathfrak{H}$ be a strongly additive function defined by

$$\mathfrak{H}(p) = \begin{cases} \lceil \mathfrak{h}(p) \rceil & \text{if } p \geqslant y \\ \mathfrak{h}(p) & \text{otherwise} \end{cases}$$

as in corollary 4.23. For an additive function g we let $\Omega(g; x) = \sum_{p \leqslant x} g(p) Z_p$. Since $\Omega(f; x) = \Omega(\mathfrak{g}; x) + \Omega(\mathfrak{h}; x)$ and $\mathfrak{H} \geqslant \mathfrak{h} \geqslant 0$, $Z_p \geqslant 0$ we have

$$\Omega(\mathfrak{g}; x) + \Omega(\mathfrak{h}; y) \leqslant \Omega(f; x) \leqslant \Omega(\mathfrak{g}; x) + \Omega(\mathfrak{H}; x)$$

Therefore, for all $t \in \mathbb{R}$,

$$\mathbb{P}_{\mathcal{F}_x}\left(\Omega(\mathfrak{g}; x) + \Omega(\mathfrak{h}; y) \geqslant t\right) \leqslant \mathbb{P}_{\mathcal{F}_x}\left(\Omega(f; x) \geqslant t\right) \leqslant \mathbb{P}_{\mathcal{F}_x}\left(\Omega(\mathfrak{g}; x) + \Omega(\mathfrak{H}; x) \geqslant t\right)$$

Now set $t := \xi_f(x; \Delta) = \mu(f; x) + \Delta\sigma(f; x)$. Our goal is to show that both the upper and lower bound are asymptotic to the (asymptotic) expression given in proposition 4.17. We will carry out the proof only for the upper bound, because the proof for the lower bound is almost identical. Let $\nu = \nu_f(x; \Delta)$. By lemma 4.7 the parameter ν is bounded when $1 \leqslant \Delta \leqslant c\sigma(f; x)$. Denote by $\delta > 0$ a real such that $0 \leqslant \nu \leqslant \delta$ uniformly in $1 \leqslant \Delta \leqslant c\sigma(f; x)$. Finally set $s := \nu + it = \nu_f(x; \Delta) + it$. By assumptions

$$\mathbb{E}_{\mathcal{F}_x}\left[e^{s\Omega(\mathfrak{g}; x) + s\Omega(\mathfrak{H}; x)}\right] = \mathbb{E}_{\mathcal{F}_x}\left[e^{s\Omega(\mathfrak{g}; x)}\right] \cdot \prod_{p \leqslant x}\left(1 - \frac{1}{p} + \frac{e^{s\mathfrak{H}(p)}}{p}\right) + O\left(\mathcal{E}(x; \nu)\right) \tag{4.63}$$

where $\mathcal{E}(x; \nu) = (\log x)^{\hat{\Psi}(f; \nu) - 3/2}$. Note that because $Z_p \in \{0; 1\}$ and $\Omega(\mathfrak{g}; x) \in \mathbb{Z}$ all the values taken by $\Omega(\mathfrak{g}; x) + \Omega(\mathfrak{H}; x)$ lie in the set $\mathbb{N} + \mathcal{D}_{\mathfrak{h}}(y)$ where $\mathcal{D}_{\mathfrak{h}}(y)$ is the set of fractional parts $\{\sum_{p \leqslant y} \mathfrak{H}(p)\varepsilon_p\} = \{\sum_{p \leqslant y} \mathfrak{h}(p)\varepsilon_p\}$, $\varepsilon_i \in \{0; 1\}$ (if it seems strange that in the fractional part we consider only the primes $p \leqslant y$ then recall that by definition $\mathfrak{H}(p) \in \mathbb{Z}$ for $p > y$). In particular $|\mathcal{D}_{\mathfrak{h}}(y)| \leqslant 2^{\pi(y)} \leqslant 2^y \ll (\log\log x)^{1/8}$ (because $y \leqslant (1/8)\log\log\log x$). Therefore we can write

$$\mathbb{E}_{\mathcal{F}_x}\left[e^{s\Omega(\mathfrak{g}; x) + s\Omega(\mathfrak{H}; x)}\right] = \sum_{\omega \in \mathbb{N} + \mathcal{D}_{\mathfrak{h}}(y)} \mathbb{P}_{\mathcal{F}_x}\left(\Omega(\mathfrak{g}; x) + \Omega(\mathfrak{H}; x) = \omega\right) \cdot e^{s\omega} \tag{4.64}$$

And this sum converges because it's finite. In the same vein the main term in (4.63) is the Laplace transform of the distribution function

$$F(x; t) = \sum_{k \in \mathbb{Z}} \mathbb{P}_{\mathcal{F}_x}\left(\Omega(\mathfrak{g}; x) = k\right) \cdot \mathbb{P}_{\mathcal{I}}\left(\sum_{p \leqslant x} \mathfrak{H}(p) X_p \leqslant t - k\right) \tag{4.65}$$

Here the X_p are independent Bernoulli random variable over some probability space $(\Theta; \mathcal{I})$. Their distribution is given by $\mathbb{P}(X_p = 1) = 1/p$ and $\mathbb{P}(X_p = 0) = 1 - 1/p$. All the "jumps" of $F(x; t)$ are contained in the set $\mathbb{N} + \mathcal{D}_{\mathfrak{h}}(y)$. Thus the main term in (4.63) admits an expansion $\sum_{\omega\in\mathbb{N}+\mathcal{D}_{\mathfrak{h}}}\delta_\omega\cdot e^{s\omega}$ similar to the one in (4.64). Consider the "kernel"

$$\mathfrak{K}(s;t)=\sum_{\substack{\omega\in\mathbb{N}+\mathcal{D}_{\mathfrak{h}}(y)\\ \omega\geqslant t}}e^{-s\omega}=\sum_{\substack{k\in\mathbb{Z}\\ k\geqslant t-2}}e^{-sk}\sum_{\substack{d\in\mathcal{D}_{\mathfrak{h}}(y)\\ k+d\geqslant t}}e^{-sd}$$

which is in modulus bounded by $\ll(1/\nu)e^{-\nu t}\cdot|\mathcal{D}_{\mathfrak{h}}(y)|\ll(1/\nu)e^{-\nu t}\cdot(\log\log x)^{1/8}$. Let $\xi:=\xi_f(x;\Delta)$. We will keep this abbreviation in use. Multiplying the left hand side of (4.63) by $\mathfrak{K}(s;\xi)$ and integrating with $(2\log\log x)^{-1}\int_{-\log\log x}^{\log\log x}\ldots dt$ we obtain

$$\sum_{\omega\in\mathbb{N}+\mathcal{D}_{\mathfrak{h}}(y)}\mathbb{P}_{\mathcal{F}_x}(\Omega(\mathfrak{g};x)+\Omega(\mathfrak{H};x)=\omega)\sum_{\omega'\in\mathbb{N}+\mathcal{D}_{\mathfrak{h}}(y),\omega'\geqslant\xi}\frac{e^{\nu(\omega-\omega')}}{2\log\log x}\int_{-\log\log x}^{\log\log x}e^{it(\omega-\omega')}dt$$

If $\omega\neq\omega'$ (with $\omega,\omega'\in\mathbb{N}+\mathcal{D}_{\mathfrak{h}}(y)$) then by our choice of y we have $1-(\log\log x)^{-1/8}\geqslant|\{\omega\}-\{\omega'\}|\geqslant(\log\log x)^{-1/8}$. It follows that $(2\log\log x)^{-1}\int_{-\log\log x}^{\log\log x}e^{it(\omega-\omega')}dt=\mathbb{I}_{\omega=\omega'}+O((\log\log x)^{-7/8})$. (Here $\mathbb{I}_{\omega=\omega'}$ is the indicator function of $\omega=\omega'$). Hence the previous equation simplifies to

$$\mathbb{P}_{\mathcal{F}_x}(\Omega(\mathfrak{g};x)+\Omega(\mathfrak{H};x)\geqslant\xi)+O(\mathbb{E}_{\mathcal{F}_x}\left[e^{\nu(\Omega(\mathfrak{g};x)+\Omega(\mathfrak{H};x))}\right](1/\nu)e^{-\nu\xi}\cdot|\mathcal{D}_{\mathfrak{h}}(y)|\cdot(\log\log x)^{-7/8})$$

and by our assumptions and lemma 4.8 the error term is $\ll(1/\nu)(\log x)^{A(f;\nu)}\cdot(\log\log x)^{-3/4}$, where as usual $A(f;\nu)=\widehat{\Psi}(f;\nu)-\nu\widehat{\Psi}'(f;\nu)-1$. Similarly, multiplying the right hand side of (4.63) by $\mathfrak{K}(s;\xi)$ and then integrating over $(2\log\log x)\int_{-\log\log x}^{\log\log x}\ldots dt$ gives

$$1-F(x;\xi)+O((1/\nu)(\log x)^{A(f;\nu)}\cdot(\log\log x)^{-3/4})$$

(Where $F(x;t)$ is defined by (4.65)). Therefore multiplying both sides of (4.63) by $\mathfrak{K}(s;\xi)$ and integrating as we've done before, we obtain the equality

$$\mathbb{P}_{\mathcal{F}_x}(\Omega(\mathfrak{g};x)+\Omega(\mathfrak{H};x)\geqslant\xi)=1-F(x;\xi)+O((1/\nu)(\log x)^{A(f;\nu)}\cdot(\log\log x)^{-3/4})\tag{4.66}$$

Note that the error term is negligible compared to the (expected) size of the main. Thus in view of (4.66) and our earlier remark it remains to estimate $1-F(x;\xi)$. To ease notation let $\Omega_X(\mathfrak{H};x)=\sum_{p\leqslant x}\mathfrak{H}(p)X_p$ where the X_p are independent Bernoulli random variables over the probability space $(\Theta;\mathcal{I})$. Rewriting (4.65) and using Cauchy's formula, we obtain

$$\begin{aligned}1-F(x;\xi)&=\sum_{k\in\mathbb{Z}}\mathbb{P}_{\mathcal{F}_x}(\Omega(\mathfrak{g};x)=\lfloor\xi\rfloor-k)\cdot\mathbb{P}_{\mathcal{I}}(\Omega_X(\mathfrak{H};x)\geqslant\{\xi\}+k)\\&=\sum_{k\in\mathbb{Z}}\left[\int_{-\pi}^{\pi}\mathbb{E}_{\mathcal{F}_x}\left[e^{s\Omega(\mathfrak{g};x)}\right]e^{-s\lfloor\xi\rfloor+sk}\cdot\frac{dt}{2\pi}\right]\cdot\mathbb{P}_{\mathcal{I}}(\Omega_X(\mathfrak{H};x)\geqslant\{\xi\}+k)\\&=\int_{-\pi}^{\pi}\mathbb{E}_{\mathcal{F}_x}\left[e^{s\Omega(\mathfrak{g};x)}\right]e^{-s\lfloor\xi\rfloor}\cdot\sum_{k\in\mathbb{Z}}e^{sk}\mathbb{P}_{\mathcal{I}}(\Omega_X(\mathfrak{H};x)\geqslant\{\xi\}+k)\cdot\frac{dt}{2\pi}\end{aligned}\tag{4.67}$$

We massage the above expression. Let $X(\mathfrak{h}) := \sum_p \mathfrak{h}(p)X_p$. By Corollary 4.23,

$$\begin{aligned}
&\sum_{k\geqslant 0} e^{sk}\cdot \mathbb{P}_{\mathcal{I}}\left(\Omega_X\left(\mathfrak{H};x\right)\geqslant\{\xi\}+k\right)\\
= &\sum_{k\geqslant 0} e^{sk}\cdot\left[\mathbb{P}_{\mathcal{I}}\left(X(\mathfrak{h})\geqslant\{\xi\}+k\right)+O_\delta\left(\frac{e^{-(2\delta+1)k}}{(\log y)^{1/4}}\right)\right] \qquad (4.68)
\end{aligned}$$

Since $\mathrm{Re}\, s = \nu \leqslant \delta$ the error term simplifies to $O((\log y)^{-1/4})$. Also, note that for $k < 0$ we trivially have $\mathbb{P}_{\mathcal{I}}(\Omega_X(\mathfrak{H};x) \geqslant \{\xi\}+k) = 1 = \mathbb{P}_{\mathcal{I}}(X(\mathfrak{h}) \geqslant \{\xi\}+k)$. Thus the identity $\sum_{k<0} e^{sk}\mathbb{P}_{\mathcal{I}}(\Omega_X(\mathfrak{H};x)\geqslant\{\xi\}+k) = \sum_{k<0} e^{sk}\mathbb{P}_{\mathcal{I}}(X(\mathfrak{h})\geqslant\{\xi\}+k)$ holds. Adding this identity to (4.68), we obtain

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} e^{sk}\mathbb{P}_{\mathcal{I}}\left(\Omega_X(\mathfrak{H};x)\geqslant\{\xi\}+k\right) &= \sum_{k\in\mathbb{Z}} e^{sk}\mathbb{P}_{\mathcal{I}}\left(X(\mathfrak{h})\geqslant\{\xi\}+k\right)+O_\delta\left((\log y)^{-1/4}\right)\\
&= e^{-s\{\xi\}}\cdot(1/s)\mathcal{P}_{\mathfrak{h}}(\xi;s)+O_\delta\left((\log y)^{-1/4}\right)
\end{aligned}$$

Inserting the above into (4.67) yields

$$1-F(x;\xi) = \int_{-\pi}^{\pi}\mathbb{E}_{\mathcal{F}_x}\left[e^{s\Omega(\mathfrak{g};x)}\right]\frac{\mathcal{P}_{\mathfrak{h}}(\xi;s)dt}{e^{s\xi}\cdot 2\pi s}+O\left(\frac{\int_{-\pi}^{\pi}|\mathbb{E}_{\mathcal{F}_x}[e^{s\Omega(\mathfrak{g};x)}]|dt}{e^{\nu\xi}\cdot(\log y)^{1/4}}\right) \qquad (4.69)$$

Since $0\leqslant\nu\leqslant\delta$ we have $\mathbb{E}_{\mathcal{F}_x}\left[e^{s\Omega(\mathfrak{g};x)}\right]\ll_\delta(\log x)^{\mathrm{Re}(\widehat{\Psi}(f;s))-1}+(\log x)^{\widehat{\Psi}(f;\nu)-3/2}$, by assumptions, because $\mathcal{A}(s)$ is analytic hence bounded in the (bounded) region $0\leqslant\mathrm{Re}\, s\leqslant\delta$ and $|\mathrm{Im}\, s|\leqslant 2\pi$. Therefore the error term in (4.69) is bounded by

$$\frac{(\log x)^{\widehat{\Psi}(f;\nu)-1}}{(\log y)^{1/4}}\cdot e^{-\nu\xi}\int_{-\pi}^{\pi}(\log x)^{\mathrm{Re}(\widehat{\Psi}(f;s)-\widehat{\Psi}(f;\nu))}\cdot dt+(\log x)^{\widehat{\Psi}(f;\nu)-3/2}\cdot e^{-\nu\xi} \qquad (4.70)$$

Since $\Psi(f;t)$ is lattice distributed we have $\widehat{\Psi}(f;s)=\sum_{k\geqslant 0}\lambda_k e^{zk}$ with $\lambda_k\geqslant 0$ not all zero. Thus $\mathrm{Re}(\widehat{\Psi}(f;\nu+it)-\widehat{\Psi}(f;\nu))\leqslant\lambda_k(\cos(kt)-1)$ for some k with $\lambda_k>0$. Hence the integral in (4.70) is $\leqslant\int(\log x)^{\lambda_k(\cos(kt)-1)}dt\ll(\log\log x)^{-1/2}$. Thus, (4.70) is bounded by $(\log x)^{\widehat{\Psi}(f;\nu)-1}e^{-\nu\xi}(\log\log x)^{-1/2}\cdot(\log y)^{-1/4}$ which is $\ll(\log x)^{A(f;\nu)}(\log\log x)^{-1/2}(\log y)^{-1/4}$ by lemma 4.8. Furthermore we have

$$\begin{aligned}
&\int_{-\pi}^{\pi}\mathbb{E}_{\mathcal{F}_x}\left[e^{s\Omega(\mathfrak{g};x)}\right]e^{-s\xi}\cdot\frac{\mathcal{P}_{\mathfrak{h}}(\xi;s)}{s}\cdot\frac{dt}{2\pi} \qquad (4.71)\\
= &\int_{-\pi}^{\pi}\mathcal{A}(s)\,(\log x)^{\widehat{\Psi}(f;s)-1}e^{-s\xi}\cdot\frac{\mathcal{P}_{\mathfrak{h}}(\xi;s)}{s}\cdot\frac{dt}{2\pi}+O_\delta\left((1/\nu)(\log x)^{\widehat{\Psi}(f;\nu)-3/2}e^{-\nu\xi}\right)
\end{aligned}$$

because $\mathbb{E}_{\mathcal{F}_x}\left[e^{s\Omega(\mathfrak{g};x)}\right]=\mathcal{A}(s)(\log x)^{\widehat{\Psi}(f;s)-1}+O((\log x)^{\widehat{\Psi}(f;\nu)-3/2})$ by assumptions, $|\mathcal{P}_{\mathfrak{h}}(a;s)|=O_\delta(1)$ by lemma 4.25 and $\mathcal{A}(s)=O_\delta(1)$ because $s=\nu+it$ lies in a bounded domain and $\mathcal{A}(s)$ is an analytic function. By lemma 4.8 the error term in (4.71) is $\ll(1/\nu)(\log x)^{A(f;\nu)-1/2}$.

Collecting (4.66), (4.69) and (4.71) gives

$$\mathbb{P}_{\mathcal{F}_x}\left(\Omega(\mathfrak{g};x)+\Omega(\mathfrak{H};x)\geqslant\xi\right)=\int_{-\pi}^{\pi}\mathcal{A}(s)\left(\log x\right)^{\widehat{\Psi}(f;s)-1}e^{-s\xi}\frac{\mathcal{P}_{\mathfrak{h}}(\xi;s)dt}{2\pi s}+O\left(\mathrm{Err}\right) \tag{4.72}$$

where $\mathrm{Err} := (1/\nu)(\log x)^{A(f;\nu)}\cdot(\operatorname{loglog} x)^{-1/2}\cdot(\log y)^{-1/4}$ (and $A(f;\nu)=\widehat{\Psi}(f;\nu)-\nu\widehat{\Psi}'(f;\nu)-1$). By lemma 4.24 the integral in (4.72) is asymptotic to

$$\mathcal{A}(\nu)(1/\nu)\mathcal{P}_{\mathfrak{h}}(\xi_f(x;\Delta);\nu)\cdot\frac{\left(\log x\right)^{\widehat{\Psi}(f;\nu)-1-\nu\widehat{\Psi}'(f;\nu)}}{(2\pi\widehat{\Psi}''(f;\nu)\operatorname{loglog} x)^{1/2}}\cdot e^{-\nu c(f)}\cdot(1+o(1)) \tag{4.73}$$

uniformly in $(\operatorname{loglog} x)^{\varepsilon}\ll\Delta\ll\sigma(f;x)$. Since $0\leqslant\nu\leqslant\delta$ is bounded we have $\mathcal{A}(\nu)\gg_{\delta}1$ because $\mathcal{A}(\cdot)$ is continuous and non-zero on the positive real axis, and $\mathcal{P}_{\mathfrak{h}}(\xi;\nu)\gg_{\delta}1$ by lemma 4.25. Thus $\mathcal{A}(\nu)\mathcal{P}_{\mathfrak{h}}(\xi;\nu)\gg_{\delta}1$. Since in addition $\log y\longrightarrow\infty$ the error term Err in (4.72) is negligible compared to (4.73). By (4.72) and (4.73), it follows that $\mathbb{P}_{\mathcal{F}_x}(\Omega(\mathfrak{g};x)+\Omega(\mathfrak{H};x)\geqslant\xi)$ is asymptotic to (4.73). Because

$$\mathbb{P}_{\mathcal{F}_x}\left(\Omega(f;x)\geqslant\xi\right)\leqslant\mathbb{P}_{\mathcal{F}_x}\left(\Omega(\mathfrak{g};x)+\Omega(\mathfrak{H};x)\geqslant\xi\right)$$

this gives an upper bound for $\mathbb{P}_{\mathcal{F}_x}(\Omega(f;x)\geqslant\xi)$ that is "asymptotically" correct. In the same way as above we establish that $\mathbb{P}_{\mathcal{F}_x}(\Omega(\mathfrak{g};x)+\Omega(\mathfrak{h};y)\geqslant\xi)$ is asymptotic to (4.73). Since $\mathbb{P}_{\mathcal{F}_x}(\Omega(\mathfrak{g};x)+\Omega(\mathfrak{h};y)\geqslant\xi)\leqslant\mathbb{P}_{\mathcal{F}_x}(\Omega(f;x)\geqslant\xi)$ this gives a lower bound for $\mathbb{P}_{\mathcal{F}_x}(\Omega(f;x)\geqslant\xi)$ that is "asymptotically" correct. The proposition follows. □

5. An asymptotic for $\mathcal{D}_f(x;\Delta)$

The object of this section is to prove Theorem 2.8. This theorem turns out to be consequence of the three general propositions from the previous section. We break down the proof into three parts, corresponding to the case when $(\operatorname{loglog} x)^{\varepsilon}\ll\Delta\ll\sigma(f;x)$ and $\Psi(f;t)$ is not lattice distributed, the case when $(\operatorname{loglog} x)^{\varepsilon}\ll\Delta\ll\sigma(f;x)$ and $\Psi(f;t)$ is lattice distributed on $\alpha\mathbb{Z}$, and the remaining case when Δ is in the range $1\leqslant\Delta=o(\sigma(f;x))$.

5.1. $\mathcal{D}_f(x;\Delta)$ when $\Psi(f;t)$ is not lattice distributed.

Lemma 5.1. *Let $f\in\mathcal{C}$. For any given $\kappa>0$ the function $1/\Gamma(\widehat{\Psi}(f;z))$ is uniformly bounded in $\operatorname{Re} z\leqslant\kappa$.*

Proof. We have $|\widehat{\Psi}(f;z)|\leqslant\widehat{\Psi}(f;\kappa)$ for $\operatorname{Re} z\leqslant\kappa$. The function $1/\Gamma(z)$ is entire, hence bounded for $|z|\leqslant\widehat{\Psi}(f;\kappa)$. It follows that $1/\Gamma(\widehat{\Psi}(f;z))$ is bounded for $\operatorname{Re} z\leqslant\kappa$. □

Proof of Part (3) of Theorem 2.8. Consider a random variable $\Omega(f;x)$ with distribution function

$$\mathbb{P}\left(\Omega(f;x)\leqslant t\right) = (1/\lfloor x\rfloor)\sum_{\substack{n\leqslant x\\ f(n)\leqslant t}}1 \tag{5.1}$$

Since $f \in \mathcal{C}$, by the mean-value theorem of Proposition 4.1

$$\begin{aligned}\mathbb{E}\left[e^{s\Omega(f;x)}\right] &= \frac{1}{\lfloor x \rfloor}\sum_{n\leqslant x} e^{sf(n)} \\ &= \frac{L(f;s)}{\Gamma(\widehat{\Psi}(f;s))}\cdot(\log x)^{\widehat{\Psi}(f;s)-1} + O\left((\log x)^{\widehat{\Psi}(f;\kappa)-3/2}\right) \qquad (5.2)\end{aligned}$$

uniformly in $0 \leqslant \kappa := \mathrm{Re}\, s \leqslant C$, $|\mathrm{Im}\, s| \leqslant \log\log x$, for any given $C > 0$. By lemma 4.4 the function $L(f;s)$ is entire, and bounded by $L(f;s) = O_{C,\varepsilon}(1 + |\mathrm{Im}\, s|^{\varepsilon})$ uniformly in $0 \leqslant \mathrm{Re}\, s \leqslant C$. Furthermore, from the product representation for $L(f;z)$, it is clear that $L(f;x)$ does not vanish for any $x \geqslant 0$. The same properties hold true for $1/\Gamma(\widehat{\Psi}(f;s))$. Indeed, by lemma 4.2 the function $\widehat{\Psi}(f;s)$ is entire, hence $1/\Gamma(\widehat{\Psi}(f;s))$ is. All the zeroes of $1/\Gamma(s)$ are located in $\mathrm{Re}\, s \leqslant 0$. Hence $1/\Gamma(\widehat{\Psi}(f;x))$ does not vanish, because $\widehat{\Psi}(f;x) \geqslant \widehat{\Psi}(f;0) > 0$ for $x \geqslant 0$. Finally by Lemma 5.1 the function $1/\Gamma(\widehat{\Psi}(f;s))$ is uniformly bounded in $\mathrm{Re}\, s \leqslant C$, for any given $C > 0$. It follows that the product

$$\mathcal{A}(s) := \frac{L(f;s)}{\Gamma(\widehat{\Psi}(f;s))}$$

is entire, non-vanishing on the positive real line, and $\mathcal{A}(s) = O_{C,\varepsilon}(1 + |\mathrm{Im}\, s|^{\varepsilon})$ uniformly in $0 \leqslant \mathrm{Re}\, s \leqslant C$, for any given $C > 0$. In addition (5.2) holds. Hence our second "general result" – proposition 4.10 – applies, and we obtain that uniformly in $(\log\log x)^{\varepsilon} \ll \Delta \leqslant c\sigma(f;x)$,

$$\mathbb{P}\left(\frac{\Omega(f;x) - \mu(f;x)}{\sigma(f;x)} \geqslant \Delta\right) \sim \frac{L(f;\nu)}{\Gamma(\widehat{\Psi}(f;\nu))}\cdot\frac{(\log x)^{\widehat{\Psi}(f;\nu)-1-\nu\widehat{\Psi}'(f;\nu)}\, e^{-\nu c(f)}}{\nu(2\pi\widehat{\Psi}''(f;\nu)\log\log x)^{1/2}}\ , \nu = \nu_f(x;\Delta)$$

By (5.1) the term on the left hand side equals to $\mathcal{D}_f(x;\Delta)$. The result follows. ☐

5.2. $\mathcal{D}_f(x;\Delta)$ **when** $\Psi(f;t)$ **is lattice distributed on** $\mathbb{Z}$. As usual when $\Psi(f;t)$ is lattice distributed we introduce the strongly additive functions $\mathfrak{g}$ and $\mathfrak{h}$ defined by

$$\mathfrak{g}(p) = \begin{cases} f(p) \text{ if } f(p) \in \mathbb{Z} \\ 0 \quad \text{otherwise}\end{cases} \quad \text{and} \quad \mathfrak{h}(p) = \begin{cases} f(p) \text{ if } f(p) \notin \mathbb{Z} \\ 0 \quad \text{otherwise}\end{cases}$$

Of course $f = \mathfrak{g} + \mathfrak{h}$. The next lemma is proved by a rather standard convolution argument (note that by lemma 4.4, we already have an asymptotic for $\sum_{n\leq x} e^{sf(n)}$).

Lemma 5.2. *Let* $f \in \mathcal{C}$. *Suppose that* $\Psi(f;t)$ *is lattice distributed on* $\mathbb{Z}$. *Given* $C > 0$, *uniformly in* $0 \leqslant \kappa := \mathrm{Re}\, s \leqslant C, |\mathrm{Im}\, s| \leqslant \log\log x$ *and strongly additive functions* $\mathfrak{H}(\cdot)$ *such that* $0 \leqslant \mathfrak{H}(p) \leqslant \lceil \mathfrak{h}(p) \rceil$,

$$\frac{1}{x}\sum_{n\leqslant x} e^{s\mathfrak{g}(n)+s\mathfrak{H}(n)} = \frac{1}{x}\sum_{n\leqslant x} e^{s\mathfrak{g}(n)}\cdot\prod_{p\leqslant x}\left(1 + \frac{e^{s\mathfrak{H}(p)}-1}{p}\right) + O_C\left((\log x)^{\widehat{\Psi}(f;\kappa)-3/2}\right)$$

Furthermore, for any given $C > 0$*, uniformly in* $0 \leqslant \kappa := \operatorname{Re} s \leqslant C$*,* $|\operatorname{Im} s| \leqslant 2\pi$*,*

$$\frac{1}{x}\sum_{n\leqslant x} e^{s\mathfrak{g}(n)} = \frac{L(\mathfrak{g}; s)}{\Gamma(\widehat{\Psi}(f; s))} \cdot (\log x)^{\widehat{\Psi}(f;s)-1} + O_C\left((\log x)^{\widehat{\Psi}(f;\kappa)-3/2}\right)$$

Proof. Let $S(\mathfrak{h}) = \{p : \mathfrak{h}(p) \neq 0\}$. Using the definition of $\mathfrak{g}$ and $\mathfrak{h}$ we find

$$\begin{aligned}
\sum_{n\geqslant 1} \frac{e^{zf(n)}}{n^s} &= \prod_{p\notin S(\mathfrak{h})}\left(1 + \frac{e^{z\mathfrak{g}(p)}}{p^s - 1}\right) \cdot \prod_{p\in S(\mathfrak{h})}\left(1 + \frac{e^{z\mathfrak{h}(p)}}{p^s - 1}\right) \\
&= \sum_{n\geqslant 1}\frac{e^{z\mathfrak{g}(n)}}{n^s} \cdot \prod_{p\in S(\mathfrak{h})}\left(1 + \frac{e^{z\mathfrak{h}(p)} - 1}{p^s}\right)
\end{aligned}$$

Note that $\mathfrak{H}(p)$ vanishes when $\mathfrak{h}(p)$ does. Hence $\mathfrak{H}(p) = 0$ when $p \notin S(\mathfrak{h})$. Therefore we can write

$$\begin{aligned}
\sum_{n\geqslant 1} \frac{e^{z\mathfrak{g}(n)+z\mathfrak{H}(n)}}{n^s} &= \prod_{p\notin S(\mathfrak{h})}\left(1 + \frac{e^{z\mathfrak{g}(p)}}{p^s - 1}\right) \prod_{p\in S(\mathfrak{h})}\left(1 + \frac{e^{z\mathfrak{H}(p)}}{p^s - 1}\right) \\
&= \sum_{n\geqslant 1}\frac{e^{z\mathfrak{g}(n)}}{n^s} \prod_{p\in S(\mathfrak{h})}\left(1 + \frac{e^{z\mathfrak{H}(p)} - 1}{p^s}\right) \\
&= \sum_{n\geqslant 1}\frac{e^{zf(n)}}{n^s} \prod_{p\in S(\mathfrak{h})}\frac{1 + (e^{z\mathfrak{H}(p)} - 1)\cdot p^{-s}}{1 + (e^{z\mathfrak{h}(p)} - 1)\cdot p^{-s}} \\
&= \sum_{n\geqslant 1}\frac{e^{zf(n)}}{n^s} \cdot \sum_{n\geqslant 1}\frac{g(z; n)}{n^s} \qquad (5.3)
\end{aligned}$$

Here the function $g(z; n)$ is multiplicative, and given explicitly by $g(z; p^\alpha) = (-1)^\alpha \cdot (e^{z\mathfrak{h}(p)} - 1)^{\alpha-1} \cdot (e^{z\mathfrak{h}(p)} - e^{z\mathfrak{H}(p)})$. To proceed we need to make a few simple remarks about $g(z; n)$. Since $0 \leqslant \mathfrak{H}(p) \leqslant \mathfrak{h}(p) + 1$ we have for $0 \leqslant \kappa := \operatorname{Re} z$,

$$|g(z; p^\alpha)| \leqslant \left(2e^{\kappa\mathfrak{h}(p)}\right)^{\alpha-1} \cdot 2e^{\kappa(\mathfrak{h}(p)+1)} \leqslant (2e^\kappa)^\alpha \cdot e^{\kappa\mathfrak{h}(p)\alpha}$$

Hence $|g(z; n)| \leqslant (2e^\kappa)^{\Omega(n)} \cdot e^{\kappa h(n)}$ where $h(n)$ is an additive function defined by $h(p^\alpha) = \mathfrak{h}(p)\alpha$. In particular $|g(z; n)| \leqslant (2e^C)^{\Omega(n)} \cdot e^{Ch(n)}$ in the half-plane $\operatorname{Re} z \leqslant C$. Note also that $g(z; p^\alpha) = 0$ whenever $p \notin S(\mathfrak{h})$. Therefore $g(z; n) = 0$ unless all the prime factors of n belong to $S(\mathfrak{h})$. We are now ready to start the proof of the lemma. Because of (5.3),

$$\sum_{n\leqslant x} e^{z\mathfrak{g}(n)+z\mathfrak{H}(n)} = \sum_{d\leqslant x} g(z; d) \cdot \sum_{n\leqslant x/d} e^{zf(n)}$$

To evaluate the above sum we use proposition 4.1. By proposition 4.1, for any fixed $C > 0$, the above sum equals to

$$\frac{L(f;z)}{\Gamma(\widehat{\Psi}(f;z))}\sum_{d\leqslant x}\frac{g(z;d)}{d}\left(\log\frac{x}{d}\right)^{\widehat{\Psi}(f;z)-1}+O\left(\sum_{d\leqslant x}\frac{|g(z;d)|}{d}\cdot(\log x)^{\widehat{\Psi}(f;\kappa)-3/2}+\sum_{d\geqslant\sqrt{x}}\frac{|g(z;d)|}{d}\right). \tag{5.4}$$

uniformly in $0\leqslant\kappa:=\mathrm{Re}\, z\leqslant C$ and $|\mathrm{Im}\, z|\leqslant\log\log x$. We'll see in a second (see discussion after equation (5.6)) that $\sum_{d\geqslant\sqrt{x}}|g(z;d)|\cdot d^{-1}\ll(\log x)^{-1}$. As for the remaining sum in the error term, we bound $\sum_{d\leqslant x}|g(z;d)|\cdot d^{-1}$ by an Euler product, and inside the Euler product we bound $|g(z;p^\alpha)|$ by $(2e^C)^\alpha\cdot(e^{C\mathfrak{h}(p)\alpha})$. Note that the Euler product will be taken over the primes $p\in S(\mathfrak{h})$ because $g(z;d)=0$ unless all the primes factors of d are in $S(\mathfrak{h})$. Thus

$$\sum_{d\leqslant x}\frac{|g(z;d)|}{d}\leqslant\prod_{p\in S(\mathfrak{h})}\left(1+\sum_{\alpha\geqslant1}\frac{(2e^C)^\alpha\cdot e^{C\mathfrak{h}(p)\alpha}}{p^\alpha}\right)$$

Since $\mathfrak{h}(p)=o(\log p)$ (to see this: by (1.3) $f(p)=o(\log p)$ hence $\mathfrak{h}(p)=o(\log p)$) there is an constant $K:=K(C)>0$ such that the above product is bounded by $\prod_{p\in S(\mathfrak{h})}(1+K\cdot e^{C\mathfrak{h}(p)}\cdot p^{-1})$. This last product is finite by lemma 4.19. Hence the error term in (5.4) is $\ll(\log x)^{-1}+(\log x)^{\widehat{\Psi}(f;\kappa)-3/2}\ll(\log x)^{\widehat{\Psi}(f;\kappa)-3/2}$. It remains to estimate the main term in (5.4). First we rewrite the main term as

$$\frac{L(f;z)}{\Gamma(\widehat{\Psi}(f;z))}\cdot(\log x)^{\widehat{\Psi}(f;z)-1}\sum_{d\leqslant x}\frac{g(z;d)}{d}\cdot\left(1-\frac{\log d}{\log x}\right)^{\widehat{\Psi}(f;z)-1} \tag{5.5}$$

We split the sum over $d\leqslant x$ into two ranges. The range $1\leqslant d\leqslant y:=\exp\left((\log x)^{1/4}\right)$ and the remaining range $d\geqslant y$ on which we simply bound by $\sum_{d>y}|g(z;d)|\cdot d^{-1}$. In the range $d\leqslant y$ we use $(1-\log d/\log x)^{\widehat{\Psi}(f;z)-1}=1+O((\log x)^{-3/4})$, which is valid because $|\widehat{\Psi}(f;z)|\leqslant\widehat{\Psi}(f;C)$ and $\log d\ll(\log x)^{1/4}$. Thus

$$\begin{aligned}&\sum_{d\leqslant x}\frac{g(z;d)}{d}\cdot\left(1-\frac{\log d}{\log x}\right)^{\widehat{\Psi}(f;z)-1} \qquad (5.6)\\ =\;&\sum_{d\leqslant y}\frac{g(z;d)}{d}\cdot\left(1+O\left(\frac{1}{(\log x)^{3/4}}\right)\right)+O\left(\sum_{d\geqslant y}\frac{|g(z;d)|}{d}\right)\\ =\;&\prod_p\left(1+\sum_{\alpha\geqslant1}\frac{g(z;p^\alpha)}{p^\alpha}\right)+O\left(\frac{1}{(\log x)^{3/4}}\sum_{d\leqslant y}\frac{|g(z;d)|}{d}\right)+O\left(\sum_{d\geqslant y}\frac{|g(z;d)|}{d}\right)\end{aligned}$$

We bound the second error term in the exactly the same way as before, getting a bound of $O((\log x)^{-3/4})$. The third error term requires a different approach. Recall that $g(z;n)$ vanishes if not all the prime factors of n are in $S(\mathfrak{h})$. Therefore to the sum $\sum_{d\geqslant y}|g(z;d)|\cdot d^{-1}$

we can add the condition $p|d \Rightarrow p \in S(\mathfrak{h})$ without altering its value. Furthermore using the inequality $|g(z;n)| \leqslant (2e^{C})^{\Omega(n)} \cdot e^{Ch(n)}$ and then applying Cauchy-Schwarz's inequality we obtain

$$\sum_{\substack{d \geqslant y \\ p|d \Rightarrow p \in S(\mathfrak{h})}} \frac{|g(z;d)|}{d} \leqslant \Big(\sum_{p|n \Rightarrow p \in S(\mathfrak{h})} \frac{\left(4e^{2C}\right)^{\Omega(n)}}{n} \Big)^{1/2} \cdot \Big(\sum_{\substack{n \geqslant y \\ p|n \Rightarrow p \in S(\mathfrak{h})}} \frac{e^{2Ch(n)}}{n} \Big)^{1/2}$$

The first sum warps into an Euler product which is finite by lemma 4.18 (and some elementary bounding). To the second sum we apply once again a Cauchy-Schwarz inequality, thus obtaining the upper bound

$$\leqslant \Big(\sum_{p|n \Rightarrow p \in S(\mathfrak{h})} \frac{e^{4Ch(n)}}{n} \Big)^{1/2} \cdot \Big(\sum_{\substack{p|n \Rightarrow p \in S(\mathfrak{h}) \\ n \geqslant y}} \frac{1}{n} \Big)^{1/2}$$

Again the first sum can be rewritten as a (finite) Euler product. The second sum is bounded by $\ll (\log y)^{-A} \cdot \sum_{p|n \Rightarrow p \in S(\mathfrak{h})} (\log n)^{A} \cdot n^{-1} \ll (\log y)^{-A}$ where the second bound comes from the corollary to lemma 4.18. It now follows that $\sum_{d \geqslant y} |g(z;d)| \cdot d^{-1} \ll (\log y)^{-A}$. Inserting this estimate in (5.6) yields

$$\begin{aligned} \sum_{d \leqslant x} \frac{g(z;d)}{d} \cdot \left(1 - \frac{\log d}{\log x}\right)^{\widehat{\Psi}(f;z)-1} &= \prod_{p} \left(1 + \sum_{\alpha \geqslant 1} \frac{g(z;p^{\alpha})}{p^{\alpha}}\right) + O\left(\frac{1}{(\log x)^{3/4}}\right) \\ &= \prod_{p} \frac{1 + (e^{z\mathfrak{H}(p)} - 1) \cdot p^{-1}}{1 + (e^{z\mathfrak{h}(p)} - 1) \cdot p^{-1}} + O\left(\frac{1}{(\log x)^{3/4}}\right) \end{aligned}$$

In the second line we simply use the definition of $g(z;p^{\alpha})$ (see (5.3)). By lemma 4.4 and lemma 5.1 we have $L(f;z)/\Gamma(\widehat{\Psi}(f;z)) \ll_C 1 + |\operatorname{Im} z| \ll_C \log\log x$ uniformly in $0 \leqslant \operatorname{Re} z \leqslant C$, $|\operatorname{Im} z| \leqslant \log\log x$. Multiplying both sides of the above equation by $L(f;z)/\Gamma(\widehat{\Psi}(f;z))(\log x)^{\widehat{\Psi}(f;z)-1}$ gives an asymptotic for (5.5). In turn an asymptotic for (5.5) allows us to evaluate (5.4) (because (5.5) is the main term for (5.4)). Since (5.4) is equal to $(1/x)\sum_{n \leqslant x} e^{z\mathfrak{g}(n) + z\mathfrak{H}(n)}$ we conclude that

$$\begin{aligned} \frac{1}{x} \sum_{n \leqslant x} e^{z\mathfrak{g}(n) + z\mathfrak{H}(n)} &= \frac{L(f;z)}{\Gamma(\widehat{\Psi}(f;z))} \prod_{p} \frac{1 + (e^{z\mathfrak{H}(p)} - 1)p^{-1}}{1 + (e^{z\mathfrak{h}(p)} - 1)p^{-1}} \cdot (\log x)^{\widehat{\Psi}(f;z)-1} + O\left(\mathcal{E}(x;\kappa)\right) \\ &= \frac{L(\mathfrak{g};z)}{\Gamma(\widehat{\Psi}(f;z))} \prod_{p} \left(1 - \frac{1}{p} + \frac{e^{z\mathfrak{H}(p)}}{p}\right) \cdot (\log x)^{\widehat{\Psi}(f;z)-1} + O\left(\mathcal{E}(x;\kappa)\right) \end{aligned}$$

where $\mathcal{E}(x;\kappa) := (\log x)^{\widehat{\Psi}(f;\kappa)-3/2}$. The second line follows from the definition of $L(f;z)$ and the fact that $\widehat{\Psi}(f;z) = \widehat{\Psi}(\mathfrak{g};z)$ when $\Psi(f;t)$ is lattice distributed on $\mathbb{Z}$. The above formula holds uniformly in strongly additive functions $\mathfrak{H}$ such that $0 \leqslant \mathfrak{H}(p) \leqslant \lceil \mathfrak{h}(p) \rceil$. Choosing $\mathfrak{H} = 0$ yields the second claim of the lemma. Choosing $\mathfrak{H}$ arbitrary (with the restriction

$\mathfrak{H}(p) = 0$ for $p > x$) and comparing the resulting asymptotic with an asymptotic for $\sum_{n \leqslant x} e^{z\mathfrak{g}(n)}$ we obtain the first claim of the lemma. □

Proof of Part (4) of Theorem 2.8. Let $\Omega_x := [1, x] \cap \mathbb{N}$ and $\mathcal{F}_x = \mathcal{P}(\Omega_x)$, where $\mathcal{P}(\Omega_x)$ is the power-set of Ω_x. Then $(\Omega_x, \mathcal{F}_x)$ equipped with the measure $\mathbb{P}_{\mathcal{F}_x}(A) = (1/\lfloor x \rfloor) \cdot \mathrm{Card}(A)$ forms a probability space. Define the random variables

$$Z_p(n) := \begin{cases} 1 \text{ if } p | n \\ 0 \text{ otherwise} \end{cases}$$

so that

$$\mathbb{P}_{\mathcal{F}_x}\left(\sum_{p \leqslant x} f(p) Z_p(n) \geqslant t \right) = \frac{1}{\lfloor x \rfloor} \cdot \# \left\{ n \leqslant x : f(n) \geqslant t \right\} \tag{5.7}$$

By lemma 5.2, for any given $C > 0$, we have uniformly in $0 \leqslant \kappa := \mathrm{Re}\, s \leqslant C$, $|\mathrm{Im}\, s| \leqslant \log\log x$ and uniformly in strongly additive $\mathfrak{H}(\cdot)$ such that $0 \leqslant \mathfrak{H}(p) \leqslant \lceil \mathfrak{h}(p) \rceil$,

$$\begin{aligned}
\mathbb{E}_{\mathcal{F}_x}\left[e^{s\Omega(\mathfrak{g};x) + s\Omega(\mathfrak{H};x)} \right] &= \frac{1}{\lfloor x \rfloor} \sum_{n \leqslant x} e^{s\mathfrak{g}(n) + s\mathfrak{H}(n)} \\
&= \left(\frac{1}{\lfloor x \rfloor} \sum_{n \leqslant x} e^{s\mathfrak{g}(n)} \right) \prod_{p \leqslant x} \left(1 - \frac{1}{p} + \frac{e^{s\mathfrak{H}(p)}}{p} \right) + O\left(\mathcal{E}(x; \kappa) \right) \\
&= \mathbb{E}_{\mathcal{F}_x}\left[e^{s\Omega(\mathfrak{g};x)} \right] \cdot \prod_{p \leqslant x} \left(1 - \frac{1}{p} + \frac{e^{s\mathfrak{H}(p)}}{p} \right) + O\left(\mathcal{E}(x; \kappa) \right) \qquad (5.8)
\end{aligned}$$

with $\mathcal{E}(x; \kappa) := (\log x)^{\widehat{\Psi}(f;\kappa) - 3/2}$. By the same lemma

$$\mathbb{E}_{\mathcal{F}_x}\left[e^{s\Omega(\mathfrak{g};x)} \right] = \frac{L(\mathfrak{g}; s)}{\Gamma(\widehat{\Psi}(f; s))} \cdot (\log x)^{\widehat{\Psi}(f;s) - 1} + O\left(\mathcal{E}(x; \kappa) \right) \tag{5.9}$$

uniformly in $0 \leqslant \kappa := \mathrm{Re}\, s \leqslant C$ and $|\mathrm{Im}\, s| \leqslant 2\pi$. The function $G(s) := L(\mathfrak{g}; s)/\Gamma(\widehat{\Psi}(f; s))$ is entire by lemma 4.20 and lemma 4.2. In addition $G(x) \neq 0$ for $x \geqslant 0$ – on the one hand it is clear that $L(\mathfrak{g}; x) \neq 0$ for $x \geqslant 0$, just by looking at its product representation; on the other hand $\widehat{\Psi}(f; x) \geqslant \widehat{\Psi}(f; 0) = 1$ for $x \geqslant 0$, hence $1/\Gamma(\widehat{\Psi}(f; x)) \neq 0$ for $x \geqslant 0$, because $1/\Gamma(z)$ vanishes only in the $\mathrm{Re}\, z \leqslant 0$ half-plane. Thus by (5.8), (5.9) and the two properties of $G(s)$ we just mentioned, the assumptions of proposition 4.17 are satisfied. Applying proposition 4.17 we obtain the desired asymptotic for (5.7) when $t := \xi_f(x; \Delta) = \mu(f; x) + \Delta\sigma(f; x)$. □

5.3. $\mathcal{D}_f(x; \Delta)$ **when** $1 \leqslant \Delta \leqslant o(\sigma(f; x))$. The desired asymptotic for $\mathcal{D}_f(x; \Delta)$ (the one indicated in part 2 of theorem 2.8) follows in the range $(\mathrm{loglog}\, x)^{\varepsilon} \ll \Delta \leqslant o(\sigma(f; x))$ from part 3 and part 4 of theorem 2.8. There is some care needed in adapting those asymptotics to the desired form. Also the case when $\Psi(f; t)$ is lattice distributed on $\alpha\mathbb{Z}$ ($\alpha \neq 1$) requires a little bit of additional work. The two lemmata below are a preparation to handle this case.

Lemma 5.3. *Let* $f \in \mathcal{C}$. *For real* $\alpha > 0$ *define* $\nu_\alpha := \nu_{f/\alpha}(x;\Delta)$ *and* $\nu := \nu_f(x;\Delta)$. *We have* $\widehat{\Psi}^{(k)}(f/\alpha;z) = (1/\alpha)^k \cdot \widehat{\Psi}^{(k)}(f;z/\alpha)$ *and* $\nu_\alpha/\alpha = \nu$. *In particular* $\widehat{\Psi}(f/\alpha;\nu_\alpha) = \widehat{\Psi}(f;\nu)$.

Proof. Note that $\Psi(f/\alpha;t) = \Psi(f;\alpha t)$. Therefore $\widehat{\Psi}(f/\alpha;z) = \widehat{\Psi}(f;z/\alpha)$. Differentiating we obtain $\widehat{\Psi}^{(k)}(f/\alpha;z) = (1/\alpha)^k \cdot \widehat{\Psi}^{(k)}(f;z/\alpha)$. It remains to prove that $\nu_\alpha/\alpha = \nu$. By definition

$$\widehat{\Psi}'(f/\alpha;\nu_\alpha) = \widehat{\Psi}'(f/\alpha;0) + \frac{\Delta}{\sigma_\Psi(f/\alpha;x)} \cdot \widehat{\Psi}''(f/\alpha;0)$$

Note that $\sigma_\Psi(f/\alpha;x) = (1/\alpha)\sigma_\Psi(f;x)$. Thus the above formula transforms into

$$(1/\alpha)\widehat{\Psi}'(f;\nu_\alpha/\alpha) = (1/\alpha)\widehat{\Psi}'(f;0) + (1/\alpha) \cdot \frac{\Delta}{\sigma_\Psi(f;x)} \cdot \widehat{\Psi}''(f;0)$$

By definition of ν, the right hand side equals to $(1/\alpha)\widehat{\Psi}'(f;\nu)$. Thus we obtain $\Psi'(f;\nu_\alpha/\alpha) = \widehat{\Psi}'(f;\nu)$. The function $\widehat{\Psi}'(f;x)$ is strictly increasing for $x > 0$. It follows that $\nu_\alpha/\alpha = \nu$ as desired. ☐

Lemma 5.4. *Let* $\alpha > 0$ *be given and* $f \in \mathcal{C}$. *For all* $x, \Delta \geqslant 1$,

$$S_f(x;\Delta) = S_{f/\alpha}(x;\Delta)$$

Proof. Let $\nu_\alpha := \nu_{f/\alpha}(x;\Delta)$ and $\nu := \nu_1$. By the previous lemma

$$\widehat{\Psi}(f/\alpha;\nu_\alpha) - \nu_\alpha \cdot \widehat{\Psi}'(f/\alpha;\nu_\alpha) = \widehat{\Psi}(f;\nu) - \nu\widehat{\Psi}'(f;\nu)$$

Furthermore $\nu_\alpha \cdot \widehat{\Psi}''(f/\alpha;\nu_\alpha)^{1/2} = \nu \cdot \widehat{\Psi}''(f;\nu)^{1/2}$. Therefore

$$\frac{(\log x)^{\widehat{\Psi}(f/\alpha;\nu_\alpha) - \nu_\alpha\widehat{\Psi}'(f/\alpha;\nu_\alpha) - 1}}{\nu_\alpha \cdot (2\pi\widehat{\Psi}''(f/\alpha;\nu_\alpha)\log\log x)^{1/2}} = \frac{(\log x)^{\widehat{\Psi}(f;\nu) - \nu\widehat{\Psi}'(f;\nu) - 1}}{\nu \cdot (2\pi\widehat{\Psi}''(f;\nu)\log\log x)^{1/2}}$$

The right hand side equals to $S_f(x;\Delta)$, while the left hand side to $S_{f/\alpha}(x;\Delta)$. It follows that $S_f(x;\Delta) = S_{f/\alpha}(x;\Delta)$ as desired. ☐

Proof of Part (1) and Part (2) of Theorem 2.8. When $\Delta \leqslant o(\sigma(f;x))$ then by lemma 4.7 $\nu = \nu_f(x;\Delta) \asymp \Delta/\sigma_\Psi(f;x) = o(1)$. We claim that

$$L(f;\nu)e^{-\nu c(f)}/\Gamma(\widehat{\Psi}(f;\nu)) = 1 + o(1) \tag{5.10}$$
$$L(\mathfrak{g};\nu)e^{-\nu c(f)}/\Gamma(\widehat{\Psi}(f;\nu)) = 1 + o(1) \tag{5.11}$$
$$\mathcal{P}_{\mathfrak{h}}(a;\nu) = 1 + o(1) \text{ uniformly in } 0 \leqslant a \leqslant 1 \tag{5.12}$$

Let $G(z) := L(f;z)e^{-zc(f)}/\Gamma(\widehat{\Psi}(f;z))$. The function $G(z)$ is entire by lemma 4.4 and lemma 4.2. Therefore $G(\nu) = G(0) + O(\nu) = 1 + o(1)$. The same proof goes for (5.11). Recall that

$$\mathcal{P}_{\mathfrak{h}}(a;\nu) = \frac{\nu}{e^\nu - 1} + \nu\sum_{k\geqslant 0} e^{\nu(k+a)} \cdot \mathbb{P}\left(X(\mathfrak{h}) \geqslant k + a\right)$$

We have $\nu/(e^\nu - 1) = 1 + O(\nu)$. Also the sum on the right is $O(1)$ throughout $0 \leqslant \nu \leqslant 1/2$ (because by lemma 4.19, $\mathbb{E}[e^{X(\mathfrak{h})}] < \infty$ hence $\mathbb{P}(X(\mathfrak{h}) \geqslant k) \leqslant e^{-k} \cdot \mathbb{E}[e^{X(\mathfrak{h})}]$). Thus $\mathcal{P}_{\mathfrak{h}}(a;\nu) =$

$1 + O(\nu)$ uniformly throughout $0 \leqslant \nu \leqslant 1/2$. Now, if $\Psi(f;t)$ is not lattice distributed and $(\log\log x)^{\varepsilon} \ll \Delta \leqslant o(\sigma(f;x))$ then by (5.10) and part 3 of theorem 2.8

$$\mathcal{D}_f(x;\Delta) \sim \frac{L(f;\nu)\cdot e^{-\nu c(f)}}{\Gamma(\widehat{\Psi}(f;\nu))} \cdot S_f(x;\Delta) \sim S_f(x;\Delta)$$

If $\Psi(f;t)$ is lattice distributed on $\mathbb{Z}$ and $(\log\log x)^{\varepsilon} \ll \Delta \leqslant o(\sigma(f;x))$ then by (5.11), (5.12) and part 4 of theorem 2.8,

$$\mathcal{D}_f(x;\Delta) \sim \frac{L(\mathfrak{g};\nu)e^{-\nu c(f)}}{\Gamma(\widehat{\Psi}(f;\nu))} \cdot \mathcal{P}_{\mathfrak{h}}(\xi_f(x;\Delta);\nu) \cdot S_f(x;\Delta) \sim S_f(x;\Delta)$$

Now consider the case when $\Psi(f;t)$ is lattice distributed on $\alpha\mathbb{Z}$ ($\alpha \neq 1$) and Δ is in the range $(\log\log x)^{\varepsilon} \ll \Delta \leqslant o(\sigma(f;x))$. Let $\nu_\alpha := \nu_{f/\alpha}(x;\Delta)$. We reduce this case to the previous one. Note that $\mathcal{D}_f(x;\Delta) = \mathcal{D}_{f/\alpha}(x;\Delta)$ and that $\Psi(f/\alpha;t)$ is lattice distributed on $\mathbb{Z}$. Therefore, using part 4 of theorem 2.8,

$$\mathcal{D}_f(x;\Delta) = \mathcal{D}_{f/\alpha}(x;\Delta) \sim \frac{L(\mathfrak{g}_{f/\alpha};\nu_\alpha)e^{-\nu_\alpha c(f/\alpha)}}{\Gamma(\widehat{\Psi}(f/\alpha;\nu_\alpha))} \cdot \mathcal{P}_{\mathfrak{h}_{f/\alpha}}(\xi_{f/\alpha}(x;\Delta);\nu_\alpha) \cdot S_{f/\alpha}(x;\Delta)$$

By lemma 5.3, $\nu_\alpha := \nu_{f/\alpha}(x;\Delta) = \alpha\nu = o(1)$. Thus the terms on the left to $S_{f/\alpha}(x;\Delta)$ are $1+o(1)$. It follows that the right hand side in the above equation is asymptotic to $S_{f/\alpha}(x;\Delta)$. But by lemma 5.4, $S_{f/\alpha}(x;\Delta) = S_f(x;\Delta)$. Hence $\mathcal{D}_f(x;\Delta) \sim S_f(x;\Delta)$ as desired. It remains to show that $\mathcal{D}_f(x;\Delta) \sim (1/\sqrt{2\pi})\int_{\Delta}^{\infty} e^{-u^2/2}\cdot du$ when Δ is in the range $\Delta \leqslant o(\sigma(f;x)^{1/3}) = o((\log\log x)^{1/6})$. This is a consequence of proposition 4.9. Indeed, let the random variable $\Omega(f;x)$ be defined by $\mathbb{P}(\Omega(f;x) \leqslant t) = (1/\lfloor x \rfloor)\#\{n \leqslant x : f(n) \leqslant t\}$. Then, by proposition 4.1,

$$\mathbb{E}\left[e^{s\Omega(f;x)}\right] = \frac{1}{\lfloor x \rfloor}\sum_{n \leqslant x} e^{sf(n)} = \frac{L(f;s)}{\Gamma(\widehat{\Psi}(f;s))} \cdot (\log x)^{\widehat{\Psi}(f;s)-1} + O\left((\log x)^{\widehat{\Psi}(f;\kappa)-3/2}\right)$$

uniformly in $|s| \leqslant \varepsilon$ for any given $\varepsilon > 0$. Since $L(f;s)/\Gamma(\widehat{\Psi}(f;s))$ is entire (by lemma 4.4 and lemma 4.2) and non-zero at $s = 0$ proposition 4.9 is applicable. It follows that

$$\mathbb{P}\left(\frac{\Omega(f;x) - \mu(f;x)}{\sigma(f;x)} \geqslant \Delta\right) \sim \frac{1}{\sqrt{2\pi}}\int_{\Delta}^{\infty} e^{-u^2/2}\cdot du$$

uniformly in $1 \leqslant \Delta \leqslant o(\sigma(f;x)^{1/3})$. Since the left hand is equal to $\mathcal{D}_f(x;\Delta)$ we are done. $\square$

6. The "structure theorem"

We break down the proof of Theorem 1.1 into three parts corresponding to the range $1 \leqslant \Delta \leqslant o(\sigma^{\alpha})$, $1 \leqslant \Delta \leqslant o(\sigma)$ and $1 \leqslant \Delta \ll \sigma$. Throughout (just as in the statement of theorem 1.1) $\sigma := \sigma(x)$ stands for a function such that $\sigma(f;x) \sim \sigma(x) \sim \sigma(g;x)$.

6.1. **The $1 \leqslant \Delta \leqslant o(\sigma(x)^{\alpha})$ range.** We now prove Part (1) and Part (2) of Theorem 1.1. The rough idea of the proof is this: We show that $\mathcal{D}_f(x;\Delta) \sim \mathcal{D}_g(x;\Delta)$ holds in the range $1 \leqslant \Delta \leqslant o(\sigma^{\alpha})$ if and only if the first $\rho(\alpha) := \lceil (1+\alpha)/(1-\alpha) \rceil$ coefficients of some power series agree. Then we relate the equality of those coefficients to the equality of moments $\int t^k d\Psi(f;t) = \int t^k d\Psi(g;t)$ for $k = 3, 4, \ldots, \rho(\alpha)$.
Let us also note at the outset that the function we will be dealing with, namely $\omega(f;z)$ and $A(f;z)$ are respectively analytic in a neighborhood of $\mathbb{R}^{+} \cup \{0\}$ (lemma 4.7) and entire (lemma 4.2).

Lemma 6.1. *Let $f \in \mathcal{C}$. Given $\varepsilon > 0$, uniformly in $(\log\log x)^{\varepsilon} \ll \Delta \leqslant o(\sigma(f;x))$,*

$$\mathcal{D}_f(x;\Delta) \sim (1/\sqrt{2\pi}\Delta) \cdot (\log x)^{\mathcal{E}(f;\Delta/\sigma_{\Psi}(f;x))}$$

where $\mathcal{E}(f;z) := A(f;\omega(f;z))$. The functions $A(f;z)$ and $\omega(f;z)$ are defined in section 3.

Proof. Let $\nu := \nu_f(x;\Delta)$. By lemma 4.7, $\nu \sim \Delta/\sigma_{\Psi}(f;x)$ when $\Delta \leqslant o(\sigma(f;x))$, and in particular $\nu = o(1)$. Thus $\widehat{\Psi}''(f;\nu) = \widehat{\Psi}''(f;0) + o(1)$ and

$$\nu(2\pi\widehat{\Psi}''(f;\nu)\log\log x)^{1/2} \sim (\Delta/\sigma_{\Psi}(f;x)) \cdot (2\pi \cdot \widehat{\Psi}''(f;0)\log\log x)^{1/2} = \sqrt{2\pi}\Delta \tag{6.1}$$

the last equality comes from $\sigma_{\Psi}(f;x)^2 = \widehat{\Psi}''(f;0)\log\log x$. By definition of ν and $\omega(f;\cdot)$ we have $\nu = \omega(f;\Delta/\sigma_{\Psi}(f;x))$, and so

$$\widehat{\Psi}(f;\nu) - 1 - \nu\widehat{\Psi}'(f;\nu) = A(f;\nu) = \mathcal{E}(f;\Delta/\sigma_{\Psi}(f;x)) \tag{6.2}$$

By part 2 of theorem 2.8, uniformly in $(\log\log x)^{\varepsilon} \ll \Delta \leqslant o(\sigma(f;x))$,

$$\mathcal{D}_f(x;\Delta) \quad \sim \quad \frac{(\log x)^{\widehat{\Psi}(f;\nu)-1-\nu\widehat{\Psi}'(f;\nu)}}{\nu(2\pi\widehat{\Psi}''(f;\nu)\log\log x)^{1/2}}\ , \nu := \nu_f(x;\Delta)$$

By (6.1), (6.2) the right hand side is asymptotic to $(\sqrt{2\pi}\Delta)^{-1}(\log x)^{\mathcal{E}(f;\Delta/\sigma_{\Psi}(f;x))}$ ☐

We now relate the asymptotic behaviour of $\mathcal{D}_f(x;\Delta)$ to the coefficients of $\mathcal{E}(f;z) = \sum_{k\geqslant 0} a_k z^k$.

Lemma 6.2. *Let $f, g \in \mathcal{C}$. Let $\varepsilon > 0$ be given. Suppose that $\sigma_{\Psi}(f;x) = \sigma_{\Psi}(g;x)$ and denote by $\sigma_{\Psi} = \sigma_{\Psi}(x)$ a function such that $\sigma_{\Psi}(f;x) = \sigma_{\Psi}(x) = \sigma_{\Psi}(g;x)$. The asymptotic relation*

$$\mathcal{D}_f(x;\Delta) \quad \sim \quad \mathcal{D}_g(x;\Delta)$$

holds uniformly in the range $(\log\log x)^{\varepsilon} \ll \Delta \leqslant o(\sigma_{\Psi}^{\alpha})$ if and only if the first $\rho(\alpha) := \lceil (1+\alpha)/(1-\alpha) \rceil$ coefficients of $\mathcal{E}(f;z) := A(f;\omega(f;z))$ and $\mathcal{E}(g;z) := A(g;\omega(g;z))$ agree.

Proof. By lemma 6.1 $\mathcal{D}_f(x;\Delta) \sim \mathcal{D}_g(x;\Delta)$ holds uniformly in $(\log\log x)^{\varepsilon} \ll \Delta \leqslant o(\sigma_{\Psi}^{\alpha})$ if and only if

$$\log\log x \cdot (\mathcal{E}(f;\Delta/\sigma_{\Psi}) - \mathcal{E}(g;\Delta/\sigma_{\Psi})) = o(1) \tag{6.3}$$

throughout $(\log\log x)^{\varepsilon} \ll \Delta \leqslant o(\sigma_{\Psi}^{\alpha})$. Let $e(z) := \mathcal{E}(f;z) - \mathcal{E}(g;z)$ and denote by a_n the n-th coefficient in the Taylor expansion of $e(z)$ about $z = 0$.

Suppose to the contrary that (6.3) holds in $(\log\log x)^{\varepsilon} \leqslant \Delta \leqslant o(\sigma_{\Psi}^{\alpha})$ but $a_m \neq 0$ for some integer $m \leqslant \rho(\alpha)$. Let m be the first such integer. Then

$$e(\Delta/\sigma_{\Psi}) = a_m \cdot (\Delta/\sigma_{\Psi})^m \cdot (1 + O(\Delta/\sigma_{\Psi})) \tag{6.4}$$

In (6.4) choose $\Delta = \sigma_{\Psi}^{1-2/m}$. This choice of Δ is allowed (i.e we have $\Delta = o(\sigma_{\Psi}^{\alpha})$) because $\rho(1-2/m) = m - 1 < \rho(\alpha)$, hence $1 - 2/m < \alpha$ and thus $\Delta = \sigma_{\Psi}^{1-2/m} = o(\sigma_{\Psi}^{\alpha})$. With this choice of Δ by (6.4), equation (6.3) becomes $a_m \cdot (\log\log x/\sigma_{\Psi}^2) = o(1)$. Hence $a_m = o(1)$ because $\sigma_{\Psi}^2 \asymp \log\log x$. Letting $x \to \infty$ we obtain $a_m = 0$, a contradiction with our initial assumption $a_m \neq 0$.

Conversely, suppose that the first $\ell := \rho(\alpha)$ coefficients of $\mathcal{E}(f;z)$ and $\mathcal{E}(g;z)$ are equal. Thus

$$e(\Delta/\sigma_{\Psi}) = \mathcal{E}(f;\Delta/\sigma_{\Psi}) - \mathcal{E}(g;\Delta/\sigma_{\Psi}) = O((\Delta/\sigma_{\Psi})^{\ell+1}) \tag{6.5}$$

uniformly in $1 \leqslant \Delta \leqslant o(\sigma_{\Psi})$. Using (6.5) and $\sigma_{\Psi}^2 \asymp \log\log x$ we obtain for $\Delta \leqslant o(\sigma_{\Psi}^{\alpha})$,

$$\begin{aligned} \log\log x \cdot (\mathcal{E}(f;\Delta/\sigma_{\Psi}) - \mathcal{E}(g;\Delta/\sigma_{\Psi})) &\ll \log\log x \cdot (\Delta/\sigma_{\Psi})^{\ell+1} \\ &\leqslant \sigma_{\Psi}^2 \cdot o(\sigma_{\Psi}^{(\alpha-1)(\ell+1)}) = o(\sigma_{\Psi}^{2+(\alpha-1)(\ell+1)}) \end{aligned}$$

The right hand side is in fact $o(1)$ because $2 + (\alpha - 1)(\ell + 1) \leqslant 0$. By the remark right above equation (6.3) this shows that $\mathcal{D}_f(x;\Delta) \sim \mathcal{D}_g(x;\Delta)$ in $(\log\log x)^{\varepsilon} \ll \Delta \leqslant o(\sigma_{\Psi}^{\alpha})$. A quick way to check $2 + (\alpha - 1)(\ell + 1) \leqslant 0$ is the following. Note that

$$\rho(\alpha) := \left\lceil \frac{1+\alpha}{1-\alpha} \right\rceil \geqslant \frac{1+\alpha}{1-\alpha} = \frac{2}{1-\alpha} + \frac{\alpha-1}{1-\alpha} = \frac{2}{1-\alpha} - 1$$

Upon rewriting the above we find $2 + (\alpha - 1)(\rho(\alpha) + 1) \leqslant 0$ as desired. □

The next lemma is crucial.

Lemma 6.3. *Let* $f, g \in \mathcal{C}$. *Suppose that* $\widehat{\Psi}''(f;0) = \widehat{\Psi}''(g;0)$. *Let* $\alpha \in (1/3, 1)$ *be given. The first* $\rho(\alpha)$ *coefficients of* $A(f;\omega(f;z))$ *and* $A(g;\omega(g;z))$ *are equal if and only if the* $k-\mathrm{th}$ *moments* $(3 \leqslant k \leqslant \rho(\alpha))$ *of* $\Psi(f;t)$ *and* $\Psi(g;t)$ *are equal, that is*

$$\int_{\mathbb{R}} t^k d\Psi(f;t) = \int_{\mathbb{R}} t^k d\Psi(g;t) \text{ for } 3 \leqslant k \leqslant \rho(\alpha)$$

Proof. Since $\alpha > 1/3$ we have $\rho(\alpha) \geqslant 3$. We work formally with power series and write $O(z^{\ell})$ to indicate terms of order $\geqslant \ell$. Denote by a_k and b_k the coefficients in the expansion around 0 of the power series $A(f;\omega(f;z))$ and $A(g;\omega(g;z))$, respectively. Suppose that $a_k = b_k$ for $k \leqslant \ell := \rho(\alpha)$. Then

$$A(f;\omega(f;z)) = A(g;\omega(g;z)) + O(z^{\ell+1}) \tag{6.6}$$

Differentiating on both sides we obtain $-\widehat{\Psi}''(f;0)\omega(f;z) = -\widehat{\Psi}''(g;0)\omega(g;z) + O(z^{\ell})$. Dividing by $\widehat{\Psi}''(f;0) = \widehat{\Psi}''(g;0)$ on both sides, we get

$$\omega(f;z) = \omega(g;z) + O(z^{\ell})$$

Expanding $A(g;\omega(g;z))$ into a Taylor series about $\omega(f;z)$, we find that

$$\begin{aligned} A(g;\omega(g;z)) &= A(g;\omega(f;z)+(\omega(g;z)-\omega(f;z))) \\ &= A(g;\omega(f;z))+\sum_{k\geqslant 1}\frac{1}{k!}\cdot(\omega(g;z)-\omega(f;z))^k\cdot A^{(k)}(g;\omega(f;z)) \end{aligned}$$

Since $\omega(g;z)-\omega(f;z)=O(z^{\ell})$ the term $k\geqslant 2$ contribute $O(z^{2\ell})$. The term $k=1$ equals to $-\omega(f;z)\widehat{\Psi}''(g;\omega(f;z))\cdot(\omega(g;z)-\omega(f;z))$ and thus contributes $O(z^{\ell+1})$ because $\omega(f;z)=O(z)$. We conclude that

$$A(g;\omega(g;z))=A(g;\omega(f;z))+O(z^{\ell+1}) \tag{6.7}$$

Inserting (6.7) into (6.6) we obtain

$$A(f;\omega(f;z))=A(g;\omega(f;z))+O\left(z^{\ell+1}\right)$$

In this relation we substitute $z\longmapsto\omega^{-1}(f;z)$. Since $\omega^{-1}(f;z)$ is zero at $z=0$ we have $\omega^{-1}(f;z)=O(z)$. Therefore, after substitution $A(f;z)=A(g;z)+O(z^{\ell+1})$. Differentiating on both sides we obtain $z\widehat{\Psi}''(f;z)=z\widehat{\Psi}''(g;z)+O(z^{\ell})$. Upon division by z we get $\widehat{\Psi}''(f;z)=\widehat{\Psi}''(g;z)+O\left(z^{\ell-1}\right)$. Since

$$\widehat{\Psi}(f;z)=\sum_{k\geqslant 0}\int_{\mathbb{R}}t^k d\Psi(f;t)\cdot\frac{z^k}{k!}$$

and $\widehat{\Psi}''(f;z)=\widehat{\Psi}''(g;z)+O(z^{\ell-1})$ with $\ell=\rho(\alpha)$ we conclude that

$$\int_{\mathbb{R}}t^k d\Psi(f;t)=\int_{\mathbb{R}}t^k d\Psi(g;t)\text{ for }k=2,3,\dots,\rho(\alpha) \tag{6.8}$$

Conversely, let us suppose that $\int_{\mathbb{R}}t^k d\Psi(f;t)=\int_{\mathbb{R}}t^k d\Psi(g;t)$ holds for all $k=3,\dots,\rho(\alpha)$. Since in addition (by assumptions) $\widehat{\Psi}''(f;0)=\widehat{\Psi}''(g;0)$ we obtain

$$\widehat{\Psi}''(f;z)=\widehat{\Psi}''(g;z)+O\left(z^{\ell-1}\right)$$

with $\ell:=\rho(\alpha)$. Multiplying both sides by z and integrating gives $A(f;z)=A(g;z)+O(z^{\ell+1})$. Since $\omega(f;z)=O(z)$, upon substituting $z\longmapsto\omega(f;z)$ in the last relation, we obtain

$$A(f;\omega(f;z))=A(g;\omega(f;z))+O\left(z^{\ell+1}\right) \tag{6.9}$$

With this in mind, we evaluate the difference $\omega(f;z)-\omega(g;z)$. Given any $h\in\mathcal{C}$, by definition $\omega(h;z)$ equals to

$$(\widehat{\Psi}')^{-1}(\widehat{\Psi}'(h;0)+z\cdot\widehat{\Psi}''(h;0))=\sum_{k\geqslant 0}\frac{z^k\cdot\widehat{\Psi}''(h;0)^k}{k!}\left[(\widehat{\Psi}')^{-1}\right]^{(k)}\left(\widehat{\Psi}'(h;0)\right) \tag{6.10}$$

where $(\widehat{\Psi}')^{-1}$ denote the inverse function (under composition) to $\widehat{\Psi}'(h;z)$ and $f^{(k)}$ stands for the k-th derivative of f. The term $k=0$ contributes 0. The term $k=1$ contributes z, since

$$[(\widehat{\Psi}')^{-1}]^{(1)}(z) = \frac{1}{\widehat{\Psi}''(h;(\widehat{\Psi}')^{-1}(h;z))}$$

so that at $z = \widehat{\Psi}'(h;0)$ that simplifies to $1/\widehat{\Psi}''(h;0)$. However, the important point here, is that the higher derivatives $[(\widehat{\Psi}')^{-1}]^{(k)}(\widehat{\Psi}'(h;0))$ will involve only the terms $\widehat{\Psi}^{(k+1)}(h;0),\ldots,\widehat{\Psi}''(h;0)$. By assumption we have $\widehat{\Psi}^{(k)}(f;0) = \widehat{\Psi}^{(k)}(g;0)$ for $2 \leqslant k \leqslant \ell := \rho(\alpha)$ therefore the power series (6.10) taken respectively for $h = f$ and $h = g$ will agree up to the $(\ell - 1)$-th term. This gives

$$\omega(f;z) = \omega(g;z) + O\left(z^{\ell}\right) \tag{6.11}$$

Expanding $A(g;\omega(f;z))$ into a Taylor series about $\omega(g;z)$, we find that

$$\begin{aligned}
& A(g;\omega(f;z)) \\
= \; & A(g;\omega(g;z) + (\omega(f;z) - \omega(g;z)) \\
= \; & A(g;\omega(g;z)) + A'(g;\omega(g;z)) \cdot (\omega(f;z) - \omega(g;z)) + O\left((\omega(f;z) - \omega(g;z))^2\right)
\end{aligned}$$

By (6.11) the third term is bounded by $O(z^{2\ell})$, while the second term is bounded by $O(z^{\ell+1})$ because $A'(g;\omega(g;z)) = O(z)$ since $A'(g;\omega(g;0)) = A'(g;0) = 0$. It follows that $A(g;\omega(f;z)) = A(g;\omega(g;z)) + O(z^{\ell+1})$. On combining this with (6.9) we conclude that $A(f;\omega(f;z)) = A(g;\omega(g;z)) + O(z^{\ell+1})$ as desired. $\square$

Proof of Part (1) and Part (2) of Theorem 1.1. By part (1) of theorem 2.8 $\mathcal{D}_f(x;\Delta) \sim (1/\sqrt{2\pi})\int_{\Delta}^{\infty} e^{-u^2/2}du$ for Δ in the range $1 \leqslant \Delta \leqslant o(\sigma(f;x)^{1/3})$. Therefore we will always have $\mathcal{D}_f(x;\Delta) \sim \mathcal{D}_g(x;\Delta)$ uniformly in $1 \leqslant \Delta \leqslant o(\sigma^{1/3})$. This proves part (1) of theorem 1.1.

By assumptions $\sigma(f;x) \sim \sigma(g;x)$. Note that

$$\sigma^2(f;x) = \widehat{\Psi}''(f;0) \cdot \log\log x + O(1)$$

Therefore $\sigma(f;x) \sim \sigma(g;x)$ gives $\widehat{\Psi}''(f;0) = \widehat{\Psi}''(g;0)$ and also $\sigma_{\Psi}(f;x) = \sigma_{\Psi}(g;x)$ because $\sigma_{\Psi}(f;x)^2 = \widehat{\Psi}''(f;0)\log\log x$. Thus the assumptions of lemma 6.2 and lemma 6.3 are satisfied. Since $\mathcal{D}_f(x;\Delta) \sim (1/\sqrt{2\pi})\int_{\Delta}^{\infty} e^{-u^2/2}\cdot du$ when $1 \leqslant \Delta \leqslant o(\sigma^{1/3})$, the relation $\mathcal{D}_f(x;\Delta) \sim \mathcal{D}_g(x;\Delta)$ holds in the range $1 \leqslant \Delta \leqslant o(\sigma^{\alpha})$ if and only if it holds in the range $(\log\log x)^{\varepsilon} \ll \Delta \leqslant o(\sigma^{\alpha})$ $(0 \leqslant \varepsilon < 1/6)$. By lemma 6.2, $\mathcal{D}_f(x;\Delta) \sim \mathcal{D}_g(x;\Delta)$ holds in that range if and only if the first $\rho(\alpha)$ coefficients of the power-series $A(f;\omega(f;z))$ and $A(g;\omega(g;z))$ coincide. By lemma 6.3 they do coincide if and only if

$$\int_{\mathbb{R}} t^k d\Psi(f;t) = \int_{\mathbb{R}} t^k d\Psi(g;t)$$

for all $k = 3,4,\ldots,\rho(\alpha)$. This chain of if and only if's proves Part (2) of Theorem 1.1. $\square$

6.2. **The** $1 \leqslant \Delta \leqslant o(\sigma)$ **range.**

Proof of Part (3) of Theorem 1.1. One direction is clear: By Theorem 2.8, when $1 \leqslant \Delta \leqslant o(\sigma(f;x))$ the asymptotic for $\mathcal{D}_f(x;\Delta)$ depends only on $\Psi(f;t)$. Hence if $\Psi(f;t) = \Psi(g;t)$ then $\mathcal{D}_f(x;\Delta) \sim \mathcal{D}_g(x;\Delta)$ throughout $1 \leqslant \Delta \leqslant o(\sigma)$.

Now we focus on the converse direction. If $\mathcal{D}_f(x;\Delta) \sim \mathcal{D}_g(x;\Delta)$ holds throughout $1 \leqslant \Delta \leqslant o(\sigma)$ then it also holds in the smaller range $1 \leqslant \Delta \leqslant o(\sigma^{\alpha})$ for any $0 < \alpha < 1$. Hence by part 2 of Theorem 1.1,

$$\int_{\mathbb{R}} t^k d\Psi(f;t) \quad = \quad \int_{\mathbb{R}} t^k d\Psi(g;t) \tag{6.12}$$

for all $k = 3, 4, \ldots, \rho(\alpha) = \lceil (1+\alpha)/(1-\alpha) \rceil$. Letting $\alpha \to 1$ it follows that (6.12) holds for all $k \geqslant 3$. Recall that

$$\widehat{\Psi}(f;z) \quad = \quad 1 + \sum_{k \geqslant 1} \int_{\mathbb{R}} t^k d\Psi(f;t) \cdot \frac{z^k}{k!}$$

Therefore $\widehat{\Psi}(f;z) - \widehat{\Psi}(g;z) = az^2 + bz$ for some $a, b \in \mathbb{R}$. In particular

$$a^2t^4 + b^2t^2 = |\widehat{\Psi}(f;it) - \widehat{\Psi}(g;it)|^2$$

The right hand side is bounded by 4 because $|\widehat{\Psi}(f;it)| \leqslant 1$ and $|\widehat{\Psi}(g;it)| \leqslant 1$. Letting $t \to \infty$ in the above equation it follows that $a = 0 = b$. Hence $\widehat{\Psi}(f;it) = \widehat{\Psi}(g;it)$. By Fourier inversion (or using probabilistic terminology, by "uniqueness of characteristic functions") $\Psi(f;t) = \Psi(g;t)$. □

6.3. **The** $1 \leqslant \Delta \leqslant c\sigma$ **range.** We prove part 4 of theorem 1.1. We break down the proof into two cases, depending on whether $\Psi(f;t)$ is or is not lattice distributed.

6.3.1. $\Psi(f;t)$ *is not lattice distributed.*

Lemma 6.4. *Let* $f, g \in \mathcal{C}$. *Suppose that* $\sigma(f;x) \sim \sigma(g;x)$. *As usual denote by* $\sigma = \sigma(x)$ *a function such that* $\sigma(f;x) \sim \sigma(x) \sim \sigma(g;x)$. *If there is a* $\delta > 0$ *such that* $\mathcal{D}_f(x;\Delta) \sim \mathcal{D}_g(x;\Delta)$ *uniformly in* $1 \leqslant \Delta \leqslant \delta\sigma$, *then* $\mathcal{Z}\left(L(f;z)\right) = \mathcal{Z}(L(g;z))$ *where* $\mathcal{Z}(h)$ *denote the zero set of* $h(\cdot)$ *(the zeroes are counted without multiplicity).*

Proof. By assumptions $\mathcal{D}_f(x;\Delta) \sim \mathcal{D}_g(x;\Delta)$ uniformly in $1 \leqslant \Delta \leqslant \delta\sigma$. Hence by part 3 of theorem 1.1, $\Psi(f;t) = \Psi(g;t)$. Therefore $S_f(x;\Delta) = S_g(x;\Delta)$ for $x, \Delta \geqslant 0$ and $\nu_f(x;\Delta) = \nu = \nu_g(x;\Delta)$ since both depend only on $\Psi(f;t)$ and $\Psi(g;t)$. Thus by part 3 of theorem 2.8 the assumption $\mathcal{D}_f(x;\Delta) \sim \mathcal{D}_g(x;\Delta)$ simplifies to

$$L(f;\nu) \cdot e^{-\nu c(f)} \quad \sim \quad L(g;\nu) \cdot e^{-\nu c(g)} \text{ uniformly in } 1 \leqslant \Delta \leqslant \delta\sigma \tag{6.13}$$

Pick a $0 < \kappa \leqslant \delta/2$ and fix $\Delta = \kappa\sigma_{\Psi}(f;x)$ in (6.13) (since $\sigma_{\Psi}(f;x) = \sigma(f;x) + o(1)$ and $\kappa < \delta$ this is allowed). We have $\nu = \nu_f(x;\Delta) = \omega(f;\Delta/\sigma_{\Psi}(f;x)) = \omega(f;\kappa)$. Letting $x \to \infty$ in (6.13) we obtain $L(f;\omega(f;\kappa))e^{-\omega(f;\kappa)c(f)} = L(g;\omega(f;\kappa))e^{-\omega(f;\kappa)c(g)}$. Since $0 < \kappa \leqslant \delta/2$ was arbitrary and $\omega(f;x)$ is increasing (with $\omega(f;0) = 0$), the functions $L(f;z)e^{-zc(f)}$ and

$L(g;z)e^{-zc(g)}$ coincide on the interval $[0;\omega(f;\delta/2)]$. Both functions are entire by lemma 4.4. Hence by analytic continuation $L(f;z)e^{-zc(f)} = L(g;z)e^{-zc(g)}$ for all $z \in \mathbb{C}$. Since exponentials never vanish we obtain $\mathcal{Z}(L(f;z)) = \mathcal{Z}(L(g;z))$. □

Lemma 6.5. *Let* $f, g \in \mathcal{C}$. *If* $\mathcal{Z}(L(f;z)) = \mathcal{Z}(L(g;z))$ *then* $f = g$, *where* $\mathcal{Z}(h)$ *denotes the zero set of* $h(\cdot)$ *(the zeroes are counted without multiplicity).*

Proof. From the definition of $L(f;z)$ we find explicitly

$$\mathcal{Z}(L(f;z)) = \left\{\frac{(2k+1)\pi i}{f(p)} + \frac{\log(p-1)}{f(p)} \ : \ k \in \mathbb{Z} \text{ and } p \text{ prime}\right\}$$

(note that $f(p) > 0$ because $f \in \mathcal{C}$). Therefore if $\mathcal{Z}(L(f;z)) = \mathcal{Z}(L(g;z))$ then

$$\left\{\frac{(2k+1)\pi i}{g(p)} + \frac{\log(p-1)}{g(p)}\right\} = \left\{\frac{(2\ell+1)\pi i}{f(q)} + \frac{\log(q-1)}{f(q)}\right\} \tag{6.14}$$

for $k, \ell \in \mathbb{Z}$ and p, q going through the set of primes. Looking at the common zero of real part 0 and smallest imaginary part we conclude that $f(2) = g(2)$. Now, fix p an odd prime. Because of (6.14) there is a prime q such that

$$\frac{(2k+1)\pi i}{g(p)} + \frac{\log(p-1)}{g(p)} = \frac{(2\ell+1)\pi i}{f(q)} + \frac{\log(q-1)}{f(q)}$$

hence

$$\frac{f(q)}{g(p)} = \frac{2\ell+1}{2k+1} = \frac{\log(q-1)}{\log(p-1)} \tag{6.15}$$

Write $p - 1 = m^r$ with $r \geqslant 1$ maximal and m a positive integer. Necessarily $r = 2^a$ with $a \geqslant 0$, otherwise p would factorize non-trivially. Further exponentiating (6.15) we get

$$q - 1 = (p-1)^{\frac{2\ell+1}{2k+1}} = m^{r\cdot\frac{2\ell+1}{2k+1}}$$

Note that $r \cdot \frac{2\ell+1}{2k+1} \in \mathbb{N}$ since $m^{r(2\ell+1)/(2k+1)}$ is an integer and $r \geqslant 1$ was chosen maximal. Therefore $r \cdot \frac{2\ell+1}{2k+1} = 2^a \cdot \frac{2\ell+1}{2k+1}$ must be a power of two, otherwise q would factorize non-trivially. Therefore the ratio $(2\ell+1)/(2k+1)$ is a power of two, but then $\ell = k$ necessarily. By (6.15) it follows that $p = q$ and $g(p) = f(p)$. Therefore $f(p) = g(p)$ for all prime p. Hence $f = g$ since f, g are strongly additive. □

Proof of Part (4) of Theorem 1.1 when $\Psi(f;t)$ *is not lattice distributed.* One direction is clear if $f = g$ then $\mathcal{D}_f(x;\Delta) = \mathcal{D}_g(x;\Delta)$. Conversely, suppose that $\mathcal{D}_f(x;\Delta) \sim \mathcal{D}_g(x;\Delta)$ throughout $1 \leqslant \Delta \leqslant \delta\sigma$, then by lemma 6.4 the zero set of $L(f;z)$ and $L(g;z)$ coincide. Hence by lemma 6.5, $f = g$, as desired. □

6.3.2. $\Psi(f;t)$ *is lattice distributed.* Suppose that $\Psi(f;t)$ is lattice distributed on $\alpha\mathbb{Z}$ for some $\alpha > 0$. Then $\Psi(f/\alpha;t)$ is lattice distributed on $\mathbb{Z}$. Since $\mathcal{D}_f(x;\Delta) = \mathcal{D}_{f/\alpha}(x;\Delta)$ we can assume without loss of generality (for the purpose of proving Part (4) of Theorem 1.1) that $\Psi(f;t)$ is lattice distributed on $\mathbb{Z}$. To such a f we associate two strongly additive function $\mathfrak{f}$ and $\mathfrak{h}_f$ defined by

$$\mathfrak{f}(p) = \begin{cases} f(p) \text{ if } f(p) \in \mathbb{Z} \\ 0 \quad \text{otherwise} \end{cases} \quad \text{and} \quad \mathfrak{h}_f(p) = \begin{cases} f(p) \text{ if } f(p) \notin \mathbb{Z} \\ 0 \quad \text{otherwise} \end{cases}$$

In particular $f(n) = \mathfrak{f}(n) + \mathfrak{h}_f(n)$. Similarly to an additive function g we associate $\mathfrak{g}$ and $\mathfrak{h}_g$ with $\mathfrak{g}$ and $\mathfrak{h}_g$ defined in the same way as $\mathfrak{f}$ and $\mathfrak{h}_f$.

Lemma 6.6. *Let* $f, g \in \mathcal{C}$. *Suppose that* $\Psi(f;t) = \Psi(g;t)$ *and that* $\Psi(f;t)$ *is lattice distributed on* $\mathbb{Z}$. *If* $\mathcal{D}_f(x;\Delta) \sim \mathcal{D}_g(x;\Delta)$ *throughout* $1 \leqslant \Delta \leqslant \delta\sigma$ *for some* $\delta > 0$, *then,*

$$L(\mathfrak{f};\nu)\, e^{-\nu c(f)} \cdot \mathcal{P}_{\mathfrak{h}_f}(\xi_f(x;\Delta);\nu) = L(\mathfrak{g};\nu)\, e^{-\nu c(g)} \cdot \mathcal{P}_{\mathfrak{h}_g}(\xi_g(x;\Delta);\nu) + o(1) \tag{6.16}$$

uniformly throughout $1 \leqslant \Delta \leqslant \delta\sigma(x)$, *with* $\nu = \nu_f(x;\Delta) = \nu_g(x;\Delta)$.

Proof. Since $\Psi(f;t) = \Psi(g;t)$ we have $S_f = S_g$ and $\nu_f = \nu = \nu_g$. Plugging the asymptotic of part 4 of theorem 2.8 into $\mathcal{D}_f(x;\Delta) \sim \mathcal{D}_g(x;\Delta)$ and cancelling $S_f(x;\Delta) = S_g(x;\Delta)$ on both sides, we obtain (6.16) but with the right hand side multiplied by an $1 + o(1)$, instead of an error term of $o(1)$. To obtain the $o(1)$ it suffices to prove that $L(\mathfrak{g};\nu)e^{-\nu c(f)}\mathcal{P}_{\mathfrak{h}_g}(\xi_g(x;\Delta);\nu) = O(1)$. The function $L(\mathfrak{g};\nu)e^{-\nu c(f)}$ is continuous and the parameter $\nu \asymp \Delta/\sigma(f;x) = O_\delta(1)$ (because $\Delta \leqslant \delta\sigma(x)$), by lemma 4.7. Therefore $L(\mathfrak{g};\nu)e^{-\nu c(f)} = O_\delta(1)$. By lemma 4.25, $\mathcal{P}_{\mathfrak{h}_g}(\xi_f(x;\Delta);\nu) = O_\delta(1)$. The claim $L(\mathfrak{g};\nu)e^{-\nu c(f)}\mathcal{P}_{\mathfrak{h}_g}(\xi_f(x;\Delta);\nu) = O_\delta(1)$ follows. $\square$

Lemma 6.7. *Let* $f \in \mathcal{C}$. *Suppose that* $\Psi(f;t)$ *is lattice distributed on* $\mathbb{Z}$. *Given* $C > 0$, *uniformly in* $0 \leqslant \nu \leqslant C$ *we have*

$$\int_0^1 \mathcal{P}_{\mathfrak{h}_f}(a;\nu)\, da = \prod_{p:\mathfrak{h}_f(p)\neq 0} \left(1 + \frac{e^{\nu \mathfrak{h}_f(p)}}{p-1}\right) \cdot \left(1 - \frac{1}{p}\right)$$

Proof. To ease notation let $X(\mathfrak{h}_f) := \sum_p \mathfrak{h}_f(p) X_p$. By definition of $\mathcal{P}_{\mathfrak{h}_f}(a;\nu)$,

$$\begin{aligned} \int_0^1 \mathcal{P}_{\mathfrak{h}_f}(a;\nu)\, da &= \nu \sum_{k\in\mathbb{Z}} \int_0^1 e^{\nu(k+a)} \cdot \mathbb{P}(X(\mathfrak{h}_f) \geqslant k + a)\, da \\ &= \nu \sum_{k\in\mathbb{Z}} \int_k^{k+1} e^{\nu a} \cdot \mathbb{P}(X(\mathfrak{h}_f) \geqslant a)\, da \\ &= \nu \int_{\mathbb{R}} e^{\nu a} \cdot \mathbb{P}(X(\mathfrak{h}_f) \geqslant a)\, da \end{aligned}$$

where we are allowed to interchange summation and integral because all the terms involved are positive. In the above integral write $e^{\nu a} da = (1/\nu) d(e^{\nu a})$ and integrate by parts

$$\nu\int_{\mathbb{R}} e^{\nu a}\cdot\mathbb{P}\left(X\left(\mathfrak{h}_f\right)\geqslant a\right)da = \left[e^{\nu a}\mathbb{P}\left(X\left(\mathfrak{h}_f\right)\geqslant a\right)\right]_{-\infty}^{\infty}-\int_{\mathbb{R}} e^{\nu a}d\mathbb{P}\left(X\left(\mathfrak{h}_f\right)\geqslant a\right)$$

By lemma 4.19 we have $\mathbb{E}[e^{AX(\mathfrak{h})}] < +\infty$ for any fixed $A > 0$. Therefore by Chernoff's bound $\mathbb{P}(X(\mathfrak{h}) \geqslant t) \leqslant \mathbb{E}[e^{AX(\mathfrak{h})}]e^{-At}$ decays faster than any power of e^{-t}. Hence $[e^{\nu a}\mathbb{P}(X(\mathfrak{h}_f) \geqslant a]_{-\infty}^{\infty}$ vanishes (because $0 \leqslant \nu \leqslant C$). It remains to note that $-d\mathbb{P}\left(X(\mathfrak{h}_f) \geqslant a\right) = d\left(1 - \mathbb{P}\left(X\left(\mathfrak{h}_f\right) < a\right)\right) = d\mathbb{P}\left(X\right($ Thus, the second term in the above equation equals to

$$\int_{\mathbb{R}} e^{\nu a}d\mathbb{P}\left(X\left(\mathfrak{h}_f\right) < a\right) = \mathbb{E}\left[e^{\nu X(\mathfrak{h}_f)}\right]$$

the Laplace transform of $X\left(\mathfrak{h}_f\right) = \sum_p \mathfrak{h}_f\left(p\right)X_p$! By independence of the X_p,

$$\mathbb{E}\left[e^{\nu X(\mathfrak{h}_f)}\right] = \prod_p \mathbb{E}\left[e^{\nu\mathfrak{h}_f(p)X_p}\right] = \prod_p\left(1+\frac{e^{\nu\mathfrak{h}_f(p)}}{p-1}\right)\cdot\left(1-\frac{1}{p}\right)$$

When $\mathfrak{h}_f(p) = 0$ the relevant term in the product simply equals to 1, therefore we can add the condition $\mathfrak{h}_f\left(p\right) \neq 0$, in the product, without altering its value. $\square$

Lemma 6.8. *Let* $f, g \in \mathcal{C}$. *Suppose that* $\Psi(f;t) = \Psi(g;t)$ *and that* $\Psi(f;t)$ *is lattice distributed on* $\mathbb{Z}$. *If* (6.16) *holds uniformly throughout* $1 \leqslant \Delta \leqslant \delta\sigma_\Psi(f;x)$ *for some* $\delta > 0$, *then*

$$L\left(f;\kappa\right)e^{-\kappa c(f)} = L\left(g;\kappa\right)e^{-\kappa c(g)}$$

for all $\kappa > 0$ *sufficiently small.*

Proof. Let us start by remarking that since $\Psi(f;t) = \Psi(g;t)$ we have $\sigma_\Psi(f;x) = \sigma_\Psi(g;x)$. It will be convenient to denote the common value by $\sigma_\Psi(x)$. As usual denote $\nu_f(x;\Delta)$ by ν. Since $\omega(f;x)$ is increasing and $\omega(f;0) = 0$, for any sufficiently small $\kappa > 0$ we can find a λ such that $\omega(f;\lambda) = \kappa$ and $0 < \lambda < \delta$. We restrict Δ to the range $\lambda\sigma_\Psi \leqslant \Delta \leqslant \lambda\sigma_\Psi + 1/\sigma_\Psi$. In this range

$$\nu = \omega(f;\Delta/\sigma_\Psi) = \omega(f;\lambda) + O(1/\sigma_\Psi^2) = \kappa + O(1/\sigma_\Psi^2)$$

because by lemma 4.7 the function $\omega(f;z)$ is analytic in a neighborhood of $\mathbb{R}^+\cup\{0\}$. Hence by "analyticity" of $L(\mathfrak{f};\nu)e^{-\nu c(f)}$ (see lemma 4.20) and lemma 4.25,

$$\begin{aligned} L(\mathfrak{f};\nu)e^{-\nu c(f)} &= L(\mathfrak{f};\kappa)e^{-\kappa c(f)} + O(1/\sigma_\Psi^2) \\ \mathcal{P}_{\mathfrak{h}_f}\left(\xi_f(x;\Delta);\nu\right) &= \mathcal{P}_{\mathfrak{h}_f}\left(\xi_f(x;\Delta);\kappa\right) + O\left(1/\sigma_\Psi^2\right) \end{aligned}$$

Of course the same relations are valid with f replaced by g. Multiplying the two relations above, we see that when Δ is confined to $\lambda\sigma_\Psi \leqslant \Delta \leqslant \lambda\sigma_\Psi + 1/\sigma_\Psi$ we can rewrite (6.16) in the following equivalent form

$$L(\mathfrak{f};\kappa)e^{-\kappa c(f)}\mathcal{P}_{\mathfrak{h}_f}\left(\xi_f(x;\Delta);\kappa\right) = L(\mathfrak{g};\kappa)e^{-\kappa c(g)}\mathcal{P}_{\mathfrak{h}_g}\left(\xi_g\left(x;\Delta\right);\kappa\right) + o(1) \tag{6.17}$$

If the above relation was true uniformly for a common $0 \leqslant a \leqslant 1$ in place of the conceivably distinct $\xi_f(x;\Delta)$ and $\xi_g(x;\Delta)$ it would be enough to integrate the above over $0 \leqslant a \leqslant 1$, use lemma 6.7 and conclude. Unfortunately, such a simplifying device is not present, so we have to be slightly more careful. Let $b \in \mathbb{R}$ be arbitrary. Recall that $\{\xi_f(x;\Delta)\} = \{\mu(f;x) + \Delta\sigma(f;x)\}$ is $1/\sigma(f;x)$ periodic in Δ. Hence, by a change of variable and the preceding lemma

$$\begin{aligned}\int_b^{b+1/\sigma(f;x)} \mathcal{P}_{\mathfrak{h}_f}(\xi_f(x;\Delta);\kappa)d\Delta &= \frac{1}{\sigma(f;x)}\int_0^1 \mathcal{P}_{\mathfrak{h}_f}(a;\kappa)\,da \\ &= \frac{1}{\sigma(f;x)}\prod_{p:\mathfrak{h}_f(p)\neq 0}\left(1+\frac{e^{\kappa\mathfrak{h}_f(p)}}{p-1}\right)\cdot\left(1-\frac{1}{p}\right)\end{aligned}$$

By lemma 4.25, $\mathcal{P}_{\mathfrak{h}_f}(\xi_f(x;\Delta);\kappa) = O_\delta(1)$. Therefore

$$\int_b^{b+1/\sigma_\Psi} \mathcal{P}_{\mathfrak{h}_f}(\xi_f(x;\Delta);\kappa)\,d\Delta = \left(\int_b^{b+1/\sigma(f;x)} + \int_{b+1/\sigma(f;x)}^{b+1/\sigma_\Psi}\right)\mathcal{P}_{\mathfrak{h}_f}(\xi_f(x;\Delta);\kappa)\,d\Delta$$

We just computed the first integral. The second integral is bounded by $O(1)$ times the length of the interval $[b+1/\sigma(f;x); b+1/\sigma_\Psi]$. That length being $\ll 1/\sigma_\Psi^2$ the second integral is bounded by $O\left(1/\sigma_\Psi^2\right)$. Now take $b = \lambda\sigma_\Psi$ and integrate the left hand side of (6.17) over $\lambda\sigma_\Psi \leqslant \Delta \leqslant \lambda\sigma_\Psi + 1/\sigma_\Psi$. We obtain

$$\frac{L(f;\kappa)e^{-\kappa c(f)}}{\sigma(f;x)}\prod_{p:\mathfrak{h}_f(p)\neq 0}\left(1+\frac{e^{\kappa\mathfrak{h}_f(p)}}{p-1}\right)\left(1-\frac{1}{p}\right) + O\left(\sigma_\Psi^{-2}\right) = \frac{L(f;\kappa)e^{-\kappa c(f)}}{\sigma(f;x)} + O\left(\sigma_\Psi^{-2}\right)$$

The same result is true with f replaced by g. Therefore integrating (6.17) over $\lambda\sigma_\Psi \leqslant \Delta \leqslant \lambda\sigma_\Psi + 1/\sigma_\Psi$ yields

$$\frac{1}{\sigma(f;x)}\cdot L(f;\kappa)e^{-\kappa c(f)} = \frac{1}{\sigma(g;x)}\cdot L(g;\kappa)e^{-\kappa c(g)} + o\left(\frac{1}{\sigma_\Psi}\right)$$

Since $\sigma(f;x) \sim \sigma(g;x)$ and $\sigma(f;x) \sim \sigma_\Psi(f;x)$ letting $x \to \infty$ we conclude $L(f;\kappa)e^{-\kappa c(f)} = L(g;\kappa)e^{-\kappa c(g)}$. Since $\kappa > 0$ was an arbitrary, sufficiently small real number, it follows that $L(f;\kappa)e^{-\kappa c(f)} = L(g;\kappa)e^{-\kappa c(g)}$ holds for all $\kappa > 0$ sufficiently small. □

Proof of Part (4) of Theorem 1.1 when $\Psi(f;t)$ is lattice distributed. If $f = g$ then $\mathcal{D}_f(x;\Delta) = \mathcal{D}_g(x;\Delta)$ for all $x,\Delta \geqslant 1$. Conversely, suppose that $\Psi(f;t)$ is lattice distributed on $\alpha\mathbb{Z}$ and that $\mathcal{D}_f(x;\Delta) \sim \mathcal{D}_g(x;\Delta)$ for $1 \leqslant \Delta \leqslant c\sigma(x)$. Since $\mathcal{D}_f(x;\Delta) \sim \mathcal{D}_g(x;\Delta)$ also holds in $1 \leqslant \Delta \leqslant o(\sigma)$, by part 3 of theorem 1.1, we have $\Psi(f;t) = \Psi(g;t)$. Note that $\Psi(f/\alpha;t) = \Psi(f;t\alpha) = \Psi(g;t\alpha) = \Psi(g/\alpha;t)$ is lattice distributed on $\mathbb{Z}$. In addition $\mathcal{D}_{f/\alpha}(x;\Delta) = \mathcal{D}_f(x;\Delta) \sim \mathcal{D}_g(x;\Delta) = \mathcal{D}_{g/\alpha}(x;\Delta)$ holds throughout $1 \leqslant \Delta \leqslant c\alpha\sigma_\alpha(x)$ where $\sigma_\alpha(x) := (1/\alpha)\sigma(x) \sim \sigma(f/\alpha;x) \sim \sigma(g/\alpha;x)$. Thus we can assume without loss of generality that $\Psi(f;t) = \Psi(g;t)$ is lattice distributed on $\mathbb{Z}$. Hence by lemma 6.6 relation (6.16) holds and thus lemma 6.8 is applicable. By lemma 6.8, $L(f;\kappa)e^{-\kappa c(f)} = L(g;\kappa)e^{-\kappa c(g)}$ for all $\kappa > 0$ sufficiently small. By analytic continuation $L(f;z)e^{-zc(f)} = L(g;z)e^{-zc(g)}$ for all $z \in \mathbb{C}$, because $L(f;z)$ and $L(g;z)$ are entire by lemma 4.4. It follows that the zero set of $L(f;z)$ and $L(g;z)$ coincides. Thus $f = g$ by lemma 6.5. □

7. Kubilius model – Theorems 2.1 and 2.2

Let $\mathcal{A}(f;z)$ denote the function defined in theorem 2.2. Since $\mathcal{A}(f;z)$ is analytic in $\mathbb{R}^+\cup\{0\}$ and $\mathcal{A}(f;0) = 1$ we have $\mathcal{A}(f;\Delta/\sigma) = 1 + o(1)$ for $1 \leqslant \Delta \leqslant o(\sigma)$. Thus Theorem 2.1 is a consequence of Theorem 2.2. The overall strategy in our proof of theorem 2.2 is to establish an asymptotic for

$$\mathbb{P}\left(\sum_{p\leqslant x} f(p)\left[X_p - \frac{1}{p}\right] \geqslant \Delta\sigma(f;x)\right) \tag{7.1}$$

and compare it with the asymptotic for $\mathcal{D}_f(x;\Delta)$ from theorem 2.8. We will deduce an asymptotic for (7.1) from the three general propositions established in section 4. Throughout this section the X_p's will denote independent Bernoulli random variables, distributed according to

$$\mathbb{P}(X_p = 1) = 1/p \quad \text{and} \quad \mathbb{P}(X_p = 0) = 1 - 1/p$$

We break down the proof of an asymptotic for (7.1) into three cases. The proof of theorem 2.2 is in section 7.4.

7.1. $(\log\log x)^{\varepsilon} \ll \Delta \ll \sigma(f;x)$ and $\Psi(f;t)$ is lattice distributed on $\mathbb{Z}$.

Throughout write $f = \mathfrak{g} + \mathfrak{h}$ with $\mathfrak{g}$ and $\mathfrak{h}$ two strongly additive functions defined by

$$\mathfrak{g}(p) = \begin{cases} f(p) \text{ if } f(p) \in \mathbb{Z} \\ 0 \quad \text{otherwise} \end{cases} \quad \text{and} \quad \mathfrak{h}(p) = \begin{cases} f(p) \text{ if } f(p) \notin \mathbb{Z} \\ 0 \quad \text{otherwise} \end{cases}$$

Lemma 7.1. *Let $f \in \mathcal{C}$. Suppose that $\Psi(f;t)$ is lattice distributed on $\mathbb{Z}$. Define the random variable $\Omega(\mathfrak{g};x) = \sum_{p\leqslant x}\mathfrak{g}(p)X_p$. Given $C > 0$, we have uniformly in the region $-C \leqslant \kappa := \operatorname{Re} s \leqslant C$, $|\operatorname{Im} s| \leqslant 2\pi$,*

$$\mathbb{E}\left[e^{s\Omega(\mathfrak{g};x)}\right] = L(\mathfrak{g};s)\cdot e^{\gamma(\widehat{\Psi}(f;s)-1)}\cdot(\log x)^{\widehat{\Psi}(f;s)-1} + O_C\left((\log x)^{\widehat{\Psi}(f;\kappa)-3/2}\right)$$

as $x \to \infty$.

Proof. Since the X_p are independent Bernoulli random variables, we have

$$\begin{aligned}\mathbb{E}\left[e^{s\Omega(\mathfrak{g};x)}\right] &= \prod_{p\leqslant x}\mathbb{E}\left[e^{s\mathfrak{g}(p)X_p}\right] = \prod_{p\leqslant x}\left(1-\frac{1}{p}\right)\cdot\left(1+\frac{e^{s\mathfrak{g}(p)}}{p-1}\right) \\ &= \left[\prod_{p\leqslant x}\left(1-\frac{1}{p}\right)^{\widehat{\Psi}(f;s)}\left(1+\frac{e^{s\mathfrak{g}(p)}}{p-1}\right)\right]\cdot\prod_{p\leqslant x}\left(1-\frac{1}{p}\right)^{-(\widehat{\Psi}(f;s)-1)} \end{aligned} \tag{7.2}$$

The product on the right equals $e^{\gamma(\widehat{\Psi}(f;s)-1)}\cdot(\log x)^{\widehat{\Psi}(f;s)-1}\cdot\left(1+O((\log x)^{-1})\right)$ by Mertens's formula (we use that $|\widehat{\Psi}(f;s)| \leqslant \widehat{\Psi}(f;C)$). On the other hand since $x \to \infty$, lemma 4.20 is applicable and so the product on the left hand side equals $L(\mathfrak{g};s)\cdot(1+O((\log x)^{-1/2})$. Thus

$$\mathbb{E}\left[e^{s\Omega(\mathfrak{g};x)}\right] = L(\mathfrak{g};s)e^{\gamma(\widehat{\Psi}(f;s)-1)}\cdot(\log x)^{\widehat{\Psi}(f;s)-1}\cdot\left(1+O\left(1/\sqrt{\log x}\right)\right)$$

By lemma 4.20 the function $L(\mathfrak{g}; s)$ is entire. Therefore $L(\mathfrak{g}; s)$ is bounded in the region $|\operatorname{Re} s| \leqslant C$, $|\operatorname{Im} s| \leqslant 2\pi$ because this region is bounded. The function $e^{\gamma(\widehat{\Psi}(f;s)-1)}$ is bounded in $\operatorname{Re} s \leqslant C$ because of the inequality $|\widehat{\Psi}(f;s)| \leqslant \widehat{\Psi}(f; \operatorname{Re} s)$. It follows that the previous equation simplifies to

$$\mathbb{E}\left[e^{s\Omega(\mathfrak{g};x)}\right] = L(\mathfrak{g}; s) e^{\gamma(\widehat{\Psi}(f;s)-1)} \cdot (\log x)^{\widehat{\Psi}(f;s)-1} + O\left((\log x)^{\widehat{\Psi}(f;\kappa)-3/2}\right)$$

which proves the lemma. □

Lemma 7.2. *Let* $f \in \mathcal{C}$. *Suppose that* $\Psi(f;t)$ *is lattice distributed on* $\mathbb{Z}$. *Given a* $\delta, \varepsilon > 0$ *we have, uniformly in* $(\log\log x)^{\varepsilon} \ll \Delta \leqslant \delta\sigma(f;x)$,

$$\mathcal{D}_f(x;\Delta) \sim \frac{e^{-\gamma(\widehat{\Psi}(f;\nu)-1)}}{\Gamma(\widehat{\Psi}(f;\nu))} \cdot \mathbb{P}\left(\sum_{p \leqslant x} f(p)\left[X_p - \frac{1}{p}\right] \geqslant \Delta\sigma(f;x)\right) \tag{7.3}$$

where $\nu := \nu_f(x;\Delta)$.

Proof. Let X_p be independent Bernoulli random variables, distributed according to

$$\mathbb{P}(X_p = 1) = \frac{1}{p} \text{ and } \mathbb{P}(X_p = 0) = 1 - \frac{1}{p}$$

Denote by $(\Omega, \mathcal{F}, \mathbb{P})$ the underlying probability space. Given a strongly additive function g, define $\Omega(g;x) = \sum_{p \leqslant x} g(p) X_p$. Let $\mathfrak{H}$ be a strongly additive function such that $0 \leqslant \mathfrak{H}(p) \leqslant \lceil \mathfrak{h}(p) \rceil$. Note that $\mathfrak{H}(p)$ vanishes when $\mathfrak{h}(p)$ does. Thus $\mathfrak{g}$ and $\mathfrak{H}$ are "supported" on two disjoint sets of primes and as a consequence the random variables $\Omega(\mathfrak{g};x)$ and $\Omega(\mathfrak{H};x)$ are independent. Therefore

$$\begin{aligned}
\mathbb{E}\left[e^{s\Omega(\mathfrak{g};x)+s\Omega(\mathfrak{H};x)}\right] &= \mathbb{E}\left[e^{s\Omega(\mathfrak{g};x)}\right] \cdot \mathbb{E}\left[e^{s\Omega(\mathfrak{H};x)}\right] \\
&= \mathbb{E}\left[e^{s\Omega(\mathfrak{g};x)}\right] \cdot \prod_{p \leqslant x}\left(1 - \frac{1}{p} + \frac{e^{s\mathfrak{H}(p)}}{p}\right)
\end{aligned}$$

Furthermore by the previous lemma for $0 \leqslant \kappa := \operatorname{Re} s \leqslant C$ and $|\operatorname{Im} s| \leqslant 2\pi$,

$$\mathbb{E}\left[e^{s\Omega(\mathfrak{g};x)}\right] = L(\mathfrak{g}; s) e^{\gamma(\widehat{\Psi}(f;s)-1)} \cdot (\log x)^{\widehat{\Psi}(f;s)-1} + O\left((\log x)^{\widehat{\Psi}(f;\kappa)-3/2}\right)$$

By lemma 4.20 and 4.2 the functions $L(\mathfrak{g}; s)$ and $e^{\gamma(\widehat{\Psi}(f;s)-1)}$ are entire. From the product representation it is clear that $L(\mathfrak{g}; x) \neq 0$ for $x \geqslant 0$. Therefore the function $L(\mathfrak{g}; s)e^{\gamma(\widehat{\Psi}(f;s)-1)}$ is in addition non-vanishing on the positive real axis. Therefore the assumption of proposition 4.17 are satisfied. It follows that the expression

$$\mathbb{P}\left(\sum_{p \leqslant x} f(p) \cdot \left[X_p - \frac{1}{p}\right] \geqslant \Delta\sigma(f;x)\right)$$

is asymptotic to

$$L(\mathfrak{g};\nu)e^{\gamma(\widehat{\Psi}(f;\nu)-1)}\cdot(1/\nu)\mathcal{P}_{\mathfrak{h}}(\xi_f(x;\Delta);\nu)\cdot\frac{(\log x)^{\widehat{\Psi}(f;\nu)-1-\nu\widehat{\Psi}'(f;\nu)}}{(2\pi\widehat{\Psi}''(f;\nu)\operatorname{loglog}x)^{1/2}}\cdot e^{-\nu c(f)} \tag{7.4}$$

uniformly in $(\operatorname{loglog}x)^{\varepsilon}\ll\Delta\leqslant c\sigma(f;x)$ where $\nu:=\nu_f(x;\Delta)$. Furthermore, by part 4 of theorem 2.8, $\mathcal{D}_f(x;\Delta)$ is asymptotic to

$$\frac{L(\mathfrak{g};\nu)}{\Gamma(\widehat{\Psi}(f;\nu))}\cdot(1/\nu)\mathcal{P}_{\mathfrak{h}}(\xi_f(x;\Delta);\nu)\cdot\frac{(\log x)^{\widehat{\Psi}(f;\nu)-1-\nu\widehat{\Psi}'(f;\nu)}}{(2\pi\widehat{\Psi}''(f;\nu)\operatorname{loglog}x)^{1/2}}\cdot e^{-\nu c(f)} \tag{7.5}$$

throughout $(\operatorname{loglog}x)^{\varepsilon}\ll\Delta\leqslant c\sigma(f;x)$. Comparing (7.4) and (7.5) proves the lemma. ☐

7.2. $(\operatorname{loglog}x)^{\varepsilon}\ll\Delta\ll\sigma(f;x)$ and $\Psi(f;t)$ is not lattice distributed.

Lemma 7.3. *Let* $f\in\mathcal{C}$. *Let* $\Omega(f;x):=\sum_{p\leqslant x}f(p)X_p$. *Given* $C>0$, *uniformly in* $-C\leqslant\kappa:=\operatorname{Re}s\leqslant C$ *and* $|\operatorname{Im}s|\leqslant\operatorname{loglog}x$,

$$\mathbb{E}\left[e^{s\Omega(f;x)}\right] = L(f;s)e^{\gamma(\widehat{\Psi}(f;s)-1)}\cdot(\log x)^{\widehat{\Psi}(f;s)-1}+O\left((\log x)^{\widehat{\Psi}(f;\kappa)-3/2}\right)$$

as $x\to\infty$.

Proof. This is the same proof as in lemma 7.1. There is a minor twist because $L(f;s)$ is no more bounded and we use lemma 4.4 instead of lemma 4.20. We give the proof anyway. Since the X_p are independent Bernoulli random variable

$$\begin{aligned}\mathbb{E}\left[e^{s\Omega(f;x)}\right] &= \prod_{p\leqslant x}\mathbb{E}\left[e^{sf(p)X_p}\right] = \prod_{p\leqslant x}\left(1-\frac{1}{p}\right)\left(1+\frac{e^{sf(p)}}{p-1}\right)\\ &= \left[\prod_{p\leqslant x}\left(1-\frac{1}{p}\right)^{\widehat{\Psi}(f;s)}\left(1+\frac{e^{sf(p)}}{p-1}\right)\right]\cdot\prod_{p\leqslant x}\left(1-\frac{1}{p}\right)^{-(\widehat{\Psi}(f;s)-1)}\end{aligned}$$

The product on the right equals $e^{\gamma(\widehat{\Psi}(f;s)-1)}\cdot(\log x)^{\widehat{\Psi}(f;s)-1}\cdot(1+O((\log x)^{-1}))$ by Mertens's formula. On the other hand since $x\to\infty$ by lemma 4.4, the product on the left equals to $L(f;s)\cdot(1+O((\log x)^{-1}))$. Thus

$$\mathbb{E}\left[e^{s\Omega(f;x)}\right]=L(f;s)e^{\gamma(\widehat{\Psi}(f;s)-1)}\cdot(\log x)^{\widehat{\Psi}(f;s)-1}\cdot(1+O((\log x)^{-1}))$$

By lemma 4.4 we have $L(f;s)\ll_C 1+\operatorname{loglog}x$ uniformly in $|\operatorname{Im}s|\leqslant\operatorname{loglog}x$ and $|\operatorname{Re}s|\leqslant C$. The function $e^{\gamma(\widehat{\Psi}(f;s)-1)}$ is bounded in $\operatorname{Re}s\leqslant C$ because of the inequality $|\widehat{\Psi}(f;s)|\leqslant\widehat{\Psi}(f;\kappa)$. Thus the previous equation simplifies to

$$\mathbb{E}\left[e^{s\Omega(f;x)}\right]=L(f;s)e^{\gamma(\widehat{\Psi}(f;s)-1)}\cdot(\log x)^{\widehat{\Psi}(f;s)-1}+O\left((\log x)^{\widehat{\Psi}(f;\kappa)-3/2}\right)$$

(where $\kappa:=\operatorname{Re}s\leqslant C$) which is the claim. ☐

Lemma 7.4. *Let* $f \in \mathcal{C}$. *Suppose that* $\Psi(f;t)$ *is not lattice distributed. Let* $\delta, \varepsilon > 0$ *be given. We have, uniformly in* $(\log\log x)^{\varepsilon} \ll \Delta \leqslant \delta\sigma(f;x)$,

$$\mathcal{D}_f(x;\Delta) \sim \frac{e^{-\gamma(\widehat{\Psi}(f;\nu)-1)}}{\Gamma(\widehat{\Psi}(f;\nu))} \cdot \mathbb{P}\left(\sum_{p\leqslant x} f(p)\left[X_p - \frac{1}{p}\right] \geqslant \Delta\sigma(f;x)\right)$$

Here, as usual $\nu := \nu_f(x;\Delta)$.

Proof. Let $\Omega(f;x) := \sum_{p\leqslant x} f(p)X_p$. By the previous lemma for any given $C > 0$, we have uniformly in $0 \leqslant \kappa := \operatorname{Re} s \leqslant C$ and $|\operatorname{Im} s| \leqslant \log\log x$,

$$\mathbb{E}\left[e^{s\Omega(f;x)}\right] = L(f;s)e^{\gamma(\widehat{\Psi}(f;s)-1)} \cdot (\log x)^{\widehat{\Psi}(f;s)-1} + O\left((\log x)^{\widehat{\Psi}(f;\kappa)-3/2}\right)$$

By lemma 4.4 the function $L(f;s)$ is entire and $L(f;s) = O_{C,\varepsilon}(1 + |\operatorname{Im} s|^{\varepsilon})$ throughout $0 \leqslant \operatorname{Re} s \leqslant C$. From the product representation for $L(f;x)$ is it clear that $L(f;s)$ doesn't vanish on $\mathbb{R}^{+}$. The function $\exp(\gamma(\widehat{\Psi}(f;s) - 1)$ is entire by lemma 4.2, never zero, and bounded in $\operatorname{Re} s \leqslant C$, because $|\widehat{\Psi}(f;s)| \leqslant \widehat{\Psi}(f;\operatorname{Re} s)$. It follows that proposition 4.10 is applicable. Therefore

$$\mathbb{P}\left(\frac{\Omega(f;x) - \mu(f;x)}{\sigma(f;x)} \geqslant \Delta\right) \sim L(f;\nu)e^{\gamma(\widehat{\Psi}(f;\nu)-1)} \cdot \frac{(\log x)^{\widehat{\Psi}(f;\nu)-1-\nu\widehat{\Psi}'(f;\nu)}}{\nu(2\pi\widehat{\Psi}''(f;\nu)\log\log x)^{1/2}} \cdot e^{-\nu c(f)}$$

with $\nu := \nu_f(x;\Delta)$ uniformly in $(\log\log x)^{\varepsilon} \ll \Delta \leqslant c\sigma(f;x)$. On the other hand, by part 3 of theorem 2.8,

$$\mathcal{D}_f(x;\Delta) \sim \frac{L(f;\nu)}{\Gamma(\widehat{\Psi}(f;\nu))} \cdot \frac{(\log x)^{\widehat{\Psi}(f;\nu)-1-\nu\widehat{\Psi}'(f;\nu)}}{\nu(2\pi\widehat{\Psi}''(f;\nu)\log\log x)^{1/2}} \cdot e^{-\nu c(f)}$$

with $\nu := \nu_f(x;\Delta)$. On comparing the two asymptotics, the lemma follows. □

7.3. **The range** $1 \leqslant \Delta \ll (\log\log x)^{1/12}$**.**

Lemma 7.5. *Let* $f \in \mathcal{C}$. *Uniformly in* $1 \leqslant \Delta \ll (\log\log x)^{1/12}$,

$$\mathbb{P}\left(\sum_{p\leqslant x} f(p)\left[X_p - \frac{1}{p}\right] \geqslant \Delta\sigma(f;x)\right) \sim \mathcal{D}_f(x;\Delta)$$

Proof. Let $\Omega(f;x) := \sum_{p\leqslant x} f(p)X_p$. By lemma 7.3, we have

$$\mathbb{E}\left[e^{s\Omega(f;x)}\right] = L(f;s)e^{\gamma(\widehat{\Psi}(f;s)-1)} \cdot (\log x)^{\widehat{\Psi}(f;s)-1} + O\left((\log x)^{\widehat{\Psi}(f;\kappa)-3/2}\right)$$

uniformly in $|s| \leqslant \varepsilon$, for any given $\varepsilon > 0$. Since $L(f;0)e^{\gamma(\widehat{\Psi}(f;0)-1)} = 1 \neq 0$, and $L(f;z)e^{\gamma(\widehat{\Psi}(f;z)-1)}$ is entire (by lemma 4.4 and 4.2), proposition 4.9 is applicable. Therefore

$$\mathbb{P}\left(\frac{\Omega(f;x) - \mu(f;x)}{\sigma(f;x)} \geqslant \Delta\right) \sim \frac{1}{\sqrt{2\pi}}\int_{\Delta}^{\infty} e^{-u^2/2} \cdot du$$

uniformly in $1 \leqslant \Delta \leqslant o((\log\log x)^{1/6})$. On the other hand, by part 1 of theorem 2.8 we know that $\mathcal{D}_f(x;\Delta) \sim (1/\sqrt{2\pi})\int_{\Delta}^{\infty} e^{-u^2/2} \cdot du$ for $1 \leqslant \Delta \leqslant o((\log\log x)^{1/6})$. The lemma follows. $\square$

7.4. **Proof of Theorem 2.2.**

Proof of Theorem 2.2. By lemma 7.2, lemma 7.4 and lemma 7.5, theorem 2.2 holds in all cases except when $\Psi(f;t)$ is lattice distributed on $\alpha\mathbb{Z}$ with $\alpha \neq 1$. So, suppose that $\Psi(f;t)$ is lattice distributed on $\alpha\mathbb{Z}$ ($\alpha \neq 1$). Then $\Psi(f/\alpha;t)$ is lattice distributed on $\mathbb{Z}$. Hence, by our earlier work, uniformly in $1 \leqslant \Delta \leqslant c\sigma(f/\alpha;x) = (c/\alpha)\sigma(f;x)$,

$$\mathcal{D}_{f/\alpha}(x;\Delta) \sim \frac{e^{-\gamma(\widehat{\Psi}(f/\alpha;\nu_\alpha)-1)}}{\Gamma(\widehat{\Psi}(f/\alpha;\nu_\alpha))} \cdot \mathbb{P}\left(\sum_{p\leqslant x} \frac{f(p)}{\alpha}\left[X_p - \frac{1}{p}\right] \geqslant \Delta\sigma(f/\alpha;x)\right) \tag{7.6}$$

where $\nu_\alpha := \nu_{f/\alpha}(x;\Delta)$. Note that $\mathcal{D}_{f/\alpha}(x;\Delta) = \mathcal{D}_f(x;\Delta)$ and that similarly the probability term in (7.6) is invariant under multiplication by α. It remains to show that $\widehat{\Psi}(f/\alpha;\nu_\alpha) = \widehat{\Psi}(f;\nu)$ where $\nu := \nu_f(x;\Delta)$ but this follows from lemma 5.3. $\square$

8. Proof of proposition 2.4

As usual, we denote by $f(n;y)$ a truncated additive function

$$f(n;y) = \sum_{\substack{p|n \\ p \leqslant y}} f(p)$$

The following lemma is due to Barban and Vinogradov (see [4], lemma 3.2, p. 122). It improves the error term obtained by Kubilius in his theorem.

Lemma 8.1. *Let f be a strongly additive function. Let $u = \log x/\log y$. Then, uniformly in $t \in \mathbb{R}$,*

$$\frac{1}{x} \cdot \#\left\{n \leqslant x : f(n;y) \geqslant t\right\} = \mathbb{P}\left(\sum_{p\leqslant y} f(p)X_p \geqslant t\right) + O\left(u^{-u/8}\right) \tag{8.1}$$

We will use theorem 2.2 and theorem 2.8 to show that when $f \in \mathcal{C}$ and $u \asymp \log\log x$, $t \asymp \mu(f;x)$ the main term on the right hand side of (8.1) is dominating.

Proof of Proposition 2.4. In lemma 8.1 take $f \in \mathcal{C}$ and in (8.1) choose $t := \xi_f(y;\Delta)$ and $u \asymp \log\log x$. In the range $1 \leqslant \Delta \leqslant c\sigma(f;y)$ we have,

$$\mathbb{P}\left(\sum_{p\leqslant y} f(p)\left[X_p - \frac{1}{p}\right] \geqslant \Delta\sigma(f;y)\right) \geqslant \mathbb{P}\left(\sum_{p\leqslant y} f(p)\left[X_p - \frac{1}{p}\right] \geqslant c\sigma^2(f;y)\right) \tag{8.2}$$

Let $w := \nu_f(x;c\sigma(f;y))$. First of all note that $w = \omega(f;c) + o(1)$ by definition of $\omega(f;z)$ and its analyticity. Therefore $\mathcal{P}_{\mathfrak{h}}(a;w) \gg 1$ uniformly in $a \geqslant 0$ by lemma 4.25 and the remark right after the statement. Also $L(f;w)e^{-wc(f)} \neq 0$ because $L(f;x)e^{-xc(f)}$ is never zero on the positive real line. It follows by theorem 2.2 and theorem 2.8 that the right hand side of (8.2) is

$$\gg (\log x)^{\widehat{\Psi}(f;w)-1-w\widehat{\Psi}'(f;w)} \cdot (\log\log x)^{-1/2}$$

Hence (8.2) is dominating over the error term in (8.1) when $t := \xi_f(y;\Delta)$, $u \asymp \log\log x$ and Δ is allowed to vary throughout $1 \leqslant \Delta \leqslant c\sigma(f;y)$. It follows that,

$$\frac{1}{x} \cdot \# \left\{ n \leqslant x : \frac{f(n;y) - \mu(f;y)}{\sigma(f;y)} \geqslant \Delta \right\} \sim \mathbb{P}\left(\sum_{p \leqslant y} f(p) \left[X_p - \frac{1}{p} \right] \geqslant \Delta\sigma(f;y) \right)$$

uniformly in $1 \leqslant \Delta \leqslant c\sigma(f;y)$ as desired. □

9. Integers to primes

The goal of this section is to prove theorem 2.6. Throughout we will work with

$$B^2(f;x) = \sum_{p \leqslant x} \frac{f(p)^2}{p}$$

rather than with $\sigma^2(f;x)$. Of course $B^2(f;x) = \sigma^2(f;x) + O(1)$ so there is little difference between the two. Let us also define $\mathcal{D}_f^{\times}(x;\Delta)$ by

$$\mathcal{D}_f^{\times}(x;\Delta) := \frac{1}{x} \cdot \# \left\{ n \leqslant x : \frac{f(n) - \mu(f;x)}{B(f;x)} \geqslant \Delta \right\}$$

This is simply $\mathcal{D}_f(x;\Delta)$ with a different normalization.

9.1. **Large deviations for $\mathcal{D}_f^{\times}(x;\Delta)$ and $\mathbb{P}(\mathcal{Z}_\Psi(x) \geqslant t)$.** We will usually need to "adjust" some of the results taken from the literature. Our main tool will be Lagrange inversion.

Lemma 9.1. *Let $C > 0$ be given. Let $f(z)$ be analytic in $|z| \leqslant C$. Suppose that $f'(z) \neq 0$ for all $|z| \leqslant C$ and that in $|z| \leqslant C$ the function $f(z)$ vanishes only at the point $z = 0$. Then, the function g defined implicitly by $f(g(z)) = z$ is analytic in a neighborhood of 0 and its n-th coefficient a_n in the Taylor expansion about 0 is given by*

$$a_n = \frac{1}{2\pi i} \oint_\gamma \frac{\zeta f'(\zeta)}{f(\zeta)^{n+1}} d\zeta$$

where γ is a circle about 0, contained in $|\zeta| \leqslant C$. The function $g(z)$ is given by

$$g(z) = \frac{1}{2\pi i} \oint_\gamma \frac{\zeta f'(\zeta)}{f(\zeta) - z} \cdot d\zeta$$

and again γ is a circle about 0, contained in $|\zeta| \leqslant C$.

The desired asymptotic for $\mathcal{D}_f^{\times}(x;\Delta)$ is contained in Maciulis's paper ([14], lemma 1A).

Lemma 9.2. *Let* f *be an additive function. Suppose that* $0 \leqslant f(p) \leqslant O(1)$ *and that* $B(f;x) \longrightarrow \infty$ *(or equivalently* $\sigma(f;x) \to \infty$*). Let* $B^2 = B^2(f;x)$*. Uniformly in the range* $1 \leqslant \Delta \leqslant o(\sigma(f;x))$ *we have*

$$\mathcal{D}_f^{\times}(x;\Delta) \sim \exp\left(-\frac{\Delta^3}{B}\sum_{k=0}^{\infty}\frac{\lambda_f(x;k+2)}{k+3}\cdot(\Delta/B)^k\right)\int_{\Delta}^{\infty} e^{-u^2/2}\cdot\frac{du}{\sqrt{2\pi}}$$

where the coefficients $\lambda_f(x;k)$ *are defined recursively by* $\lambda_f(x;0) = 0, \lambda_f(x;1) = 1$ *and*

$$\lambda_f(x;j) = -\sum_{i=2}^{j}\frac{1}{i!}\cdot\left(\frac{1}{B^2(f;x)}\sum_{p\leqslant x}\frac{f(p)^{i+1}}{p}\right)\sum_{k_1+\ldots+k_i=j}\lambda_f(x;k_1)\cdot\ldots\cdot\lambda_f(x;k_i)$$

Further there is a constant $C = C(f)$ *such that* $|\lambda_f(x;k)| \leqslant C^k$ *for all* $k, x \geqslant 1$.

Proof. Except the bound $|\lambda_f(x;k)| \leqslant C^k$, the totality of the lemma is contained in Maciulis's paper ([14], lemma 1A). Let us prove that $|\lambda_f(x;k)| \leqslant C^k$ for a suitable positive constant $C > 0$. To do so, we consider the power series

$$\mathcal{G}_f(x;z) = \sum_{j\geqslant 2}\lambda_f(x;j)\cdot z^j + z$$

Let us look in more detail at the sum over $j \geqslant 2$. By making use of the recurrence relation for $\lambda_f(x;j)$ we see that the sum in question equals to

$$\begin{aligned}
&= -\sum_{j\geqslant 2}\sum_{i=2}^{j}\frac{1}{i!}\cdot\left(\frac{1}{B^2(f;x)}\sum_{p\leqslant x}\frac{f(p)^{i+1}}{p}\right)\sum_{k_1+\ldots+k_i=j}\lambda_f(x;k_1)\cdot\ldots\cdot\lambda_f(x;k_i)\cdot z^j\\
&= -\sum_{i\geqslant 2}\frac{1}{i!}\cdot\left(\frac{1}{B^2(f;x)}\sum_{p\leqslant x}\frac{f(p)^{i+1}}{p}\right)\cdot\left(\sum_{k\geqslant 0}\lambda_f(x;k)z^k\right)^i\\
&= -\frac{1}{B^2(f;x)}\sum_{p\leqslant x}\frac{f(p)}{p}\sum_{i\geqslant 2}\frac{1}{i!}\cdot f(p)^i\cdot\mathcal{G}_f(x;z)^i\\
&= -\frac{1}{B^2(f;x)}\sum_{p\leqslant x}\frac{f(p)}{p}\cdot\left(e^{f(p)\mathcal{G}_f(x;z)} - f(p)\mathcal{G}_f(x;z) - 1\right)
\end{aligned}$$

The above calculation reveals that $\mathcal{F}_f(x;\mathcal{G}_f(x;z)) = z$ where $\mathcal{F}_f(x;z)$ is defined by

$$\begin{aligned}
\mathcal{F}_f(x;z) &= z + \frac{1}{B^2(f;x)}\sum_{p\leqslant x}\frac{f(p)}{p}\cdot\left(e^{f(p)z} - f(p)z - 1\right)\\
&= \frac{1}{B^2(f;x)}\sum_{p\leqslant x}\frac{f(p)}{p}\cdot\left(e^{f(p)z} - 1\right)
\end{aligned}$$

Since all the $f(p)$ are bounded by some $M \geqslant 0$, we have $\mathcal{F}_f(x;z) = z + O\left(Mz^2\right)$ when z is in a neighborhood of 0, furthermore the implicit constant in the big O, depends only on M. Therefore $\mathcal{F}_f(x;z) \gg 1$ for z in the annulus $B/2 \leqslant |z| \leqslant B$, where B is a sufficiently small constant, depending only on M. Let us also note the derivative

$$\frac{d}{dz} \cdot \mathcal{F}_f(x;z) = \frac{1}{B^2(f;x)} \sum_{p \leqslant x} \frac{f(p)^2}{p} \cdot e^{f(p)z}$$

doesn't vanish and is bounded uniformly in $|z| \leqslant B$ for B sufficiently small, depending only on M. Hence by Lagrange inversion the function $\mathcal{G}_f(x;z)$ is for each $x \geqslant 1$ analytic in the neighborhood $|z| \leqslant B$ of 0, and in addition, its coefficients $\lambda_f(x;k)$ are given by

$$\lambda_f(x;k) \quad = \quad \frac{1}{2\pi i} \oint_{|\zeta|=B/2} \frac{\zeta \cdot (d/d\zeta)\mathcal{F}_f(x;\zeta)}{\mathcal{F}_f(x;\zeta)^{k+1}} d\zeta$$

However we know that $\mathcal{F}_f(x;\zeta) \gg 1$ and that $(d/d\zeta)\mathcal{G}_f(x;\zeta) \ll 1$ on the boundary $|\zeta| = B/2$ with the implicit constant depending only on M. Therefore, the integral is bounded by C^k, for some $C > 0$ depending only on M. Hence $|\lambda_f(x;k)| \leqslant C^k$. □

From Hwang's paper [11] – itself heavily based on the same methods as used by Maciulis [14] – we obtain the next lemma. Since Hwang's lemma is not exactly what is stated below, we include the deduction.

Lemma 9.3. *Let Ψ be a distribution function. Suppose that there is an $\alpha > 0$ such that $\Psi(\alpha) - \Psi(0) = 1$. Let $B^2 = B^2(x) \to \infty$ be some function tending to infinity. Uniformly in the range $1 \leqslant \Delta \leqslant o(B)$ we have*

$$\mathbb{P}\left(\mathcal{Z}_\Psi\left(B^2\right) \geqslant \Delta B\right) \sim \exp\left(-\frac{\Delta^3}{B} \sum_{k=0}^{\infty} \frac{\Lambda(\Psi;k+2)}{k+3} \cdot (\Delta/B)^k\right) \int_\Delta^\infty e^{-u^2/2} \cdot \frac{du}{\sqrt{2\pi}}$$

the coefficients $\Lambda(\Psi;k)$ satisfy $\Lambda(\Psi;0) = 0, \Lambda(\Psi;1) = 1$ and the recurrence relation

$$\Lambda(\Psi;j) = -\sum_{2 \leqslant \ell \leqslant j} \frac{1}{\ell!} \int_{\mathbb{R}} t^{\ell-1} d\Psi(f;t) \sum_{k_1+\ldots+k_\ell=j} \Lambda(\Psi;k_1) \cdot \ldots \cdot \Lambda(\Psi;k_\ell)$$

Furthermore there is a constant $C = C(\Psi) > 0$ such that $|\Lambda(\Psi;k)| \leqslant C^k$ for $k \geqslant 1$.

Proof. Let $u(z) = u(\Psi;z) = \int_{\mathbb{R}}(e^{zt} - zt - 1) \cdot t^{-2} d\Psi(t)$. Note that $u(z)$ is entire because $\Psi(t)$ is supported on a compact interval. By Hwang's theorem 1 (see [11])

$$\mathbb{P}\left(\mathcal{Z}_\Psi\left(B^2\right) \geqslant \Delta B\right) \sim \exp\left(-B^2 \sum_{k \geqslant 0} \frac{\Lambda(\Psi;k+2)}{k+3} \cdot (\Delta/B)^k\right) \int_\Delta^\infty e^{-u^2/2} \cdot \frac{du}{\sqrt{2\pi}}$$

uniformly in $1 \leqslant \Delta \leqslant o(B(x))$ with the coefficients $\Lambda(\Psi;k)$ given by $\Lambda(\Psi;0) = 0, \Lambda(\Psi;1) = 1$ and for $k \geqslant 0$,

$$\begin{aligned}\frac{\Lambda(\Psi;k+2)}{k+3} \quad &= \quad -\frac{1}{k+3} \cdot \frac{1}{2\pi i} \oint_\gamma u''(z) \cdot \left(\frac{u'(z)}{z}\right)^{-k-3} \cdot \frac{dz}{z^{k+2}} \\ &= \quad -\frac{1}{k+3} \oint_\gamma \frac{zu''(z)}{u'(z)^{k+3}} \cdot \frac{dz}{2\pi i}\end{aligned}$$

(we set $m = k+3$, $k \geqslant 0$, $q_m = \Lambda(\Psi; k+2)/(k+3)$ in equation (7) of [11] and rewrite equation (8) in [11] in terms of Cauchy's formula). Here γ is a small circle around the origin. First let us show that the coefficients $\Lambda(\Psi; k+2)$ are bounded by C^k for a sufficiently large (but fixed) $C > 0$. Around $z = 0$ we have $u'(z) = \int_{\mathbb{R}} (e^{zt} - 1) \cdot t^{-1} d\Psi(t) = z + O(z^2)$. Therefore if we choose the circle γ to have sufficiently small radius then $u'(z) \gg 1$ for z on γ. Hence looking at the previous equation, the Cauchy integral defining $\Lambda(\Psi; k+2)/k+3$ is bounded in modulus by $\ll C^{k+3}$ for some constant $C > 0$. The bound $|\Lambda(\Psi; k)| \ll C^k$ ensues (perhaps with a larger C than earlier). Our goal now is to show that $\Lambda(\Psi; k)$ satisfies the recurrence relation given in the statement of the lemma. Multiplying by ξ^{k+3} and summing over $k \geqslant 0$ we obtain

$$\begin{aligned}\sum_{k\geqslant 0} \frac{\Lambda(\Psi; k+2)}{k+3} \cdot \xi^{k+3} \;&=\; -\sum_{k\geqslant 0} \frac{\xi^{k+3}}{k+3} \oint_\gamma \frac{z u''(z)}{u'(z)^{k+3}} \cdot \frac{dz}{2\pi i} \\ = \;& -\oint_\gamma z u''(z) \sum_{k\geqslant 0} \frac{1}{k+3} \cdot \left(\frac{\xi}{u'(z)}\right)^{k+3} \cdot \frac{dz}{2\pi i} \\ = \;& \oint_\gamma z u''(z) \cdot \left(-\frac{\xi}{u'(z)} - \frac{1}{2} \cdot \frac{\xi^2}{u'(z)^2} - \log\left(1 - \frac{\xi}{u'(z)}\right)\right) \cdot \frac{dz}{2\pi i}\end{aligned}$$

Differentiating with respect to ξ on both sides yields

$$\mathcal{G}_\Psi(\xi) \;=\; \sum_{k\geqslant 0} \Lambda(\Psi; k+2)\xi^{k+2} \;=\; \oint \left(\frac{z u''(z)}{u'(z) - \xi} - \frac{z u''(z)}{u'(z)} - \frac{\xi z u''(z)}{u'(z)^2}\right) \frac{dz}{2\pi i}$$

By Lagrange inversion this last integral is equal to $(u')^{-1}(\xi) - 0 - \xi$ where $(u')^{-1}$ denotes the inverse function to $u'(z)$. Hence

$$\sum_{k\geqslant 0} \Lambda(\Psi; k)\xi^k \;=\; (u')^{-1}(\xi) \tag{9.1}$$

Let us compose this with $u'(\cdot)$ on both sides and compute the resulting left hand side. First of all we expand $u'(z)$ in a power series. This gives

$$u'(z) = \int_{\mathbb{R}} \frac{e^{zt} - 1}{t} d\Psi(t) = \sum_{\ell\geqslant 1} \frac{1}{\ell!} \int_{\mathbb{R}} t^{\ell-1} d\Psi(t) \cdot z^\ell$$

Therefore, composing (9.1) with $u'(\cdot)$ yields

$$\begin{aligned}
\xi &= u'\left(\sum_{k\geqslant 0}\Lambda(\Psi;k)\xi^k\right) = \sum_{\ell\geqslant 1}\frac{1}{\ell!}\int_{\mathbb{R}} t^{\ell-1}d\Psi(t)\cdot\left(\sum_{k\geqslant 0}\Lambda(\Psi;k)\xi^k\right)^{\ell} \\
&= \sum_{\ell\geqslant 1}\frac{1}{\ell!}\int_{\mathbb{R}} t^{\ell-1}d\Psi(t)\cdot\left(\sum_{k_1,\ldots,k_\ell\geqslant 1}\Lambda(\Psi;k_1)\cdot\ldots\cdot\Lambda(\Psi;k_\ell)\cdot\xi^{k_1+\ldots+k_\ell}\right) \\
&= \sum_{m\geqslant 1}\left(\sum_{1\leqslant\ell\leqslant m}\frac{1}{\ell!}\int_{\mathbb{R}} t^{\ell-1}d\Psi(t)\sum_{k_1+\ldots+k_\ell=m}\Lambda(\Psi;k_1)\cdot\ldots\cdot\Lambda(\Psi;k_\ell)\right)\cdot\xi^m
\end{aligned}$$

Thus the first coefficient $\Lambda(\Psi;1)$ is equal to 1, as desired, while for the terms $m\geqslant 2$ we have

$$\sum_{1\leqslant\ell\leqslant m}\frac{1}{\ell!}\int_{\mathbb{R}} t^{\ell-1}d\Psi(t)\sum_{k_1+\ldots+k_\ell=m}\Lambda(\Psi;k_1)\cdot\ldots\cdot\Lambda(\Psi;k_\ell)=0$$

The first term $\ell=1$ is equal to $\Lambda(\Psi;m)$. It suffice to move it on the right hand side of the equation, to obtain the desired recurrence relation. □

Finally we will need one last result "from the literature". Namely a weak form of the method of moments. For a proof we refer the reader to Gut's book [8], p. 237. (Note that the next lemma follows from the result in [8] because in our case the random variables are positive, and bounded, in particular their distribution is determined uniquely by their moments).

Lemma 9.4. *Let Ψ be a distribution function. Suppose that there is an $a>0$ such that $\Psi(a)-\Psi(0)=1$. Let $F(x;t)$ be a sequence of distribution functions, one for each $x>0$. If for each $k\geqslant 0$,*

$$\int_{\mathbb{R}} t^k\,dF(x;t)\longrightarrow\int_{\mathbb{R}} t^k d\Psi(t)$$

Then $F(x;t)\longrightarrow\Psi(t)$ at all continuity points t of $\Psi(t)$.

9.2. **A transfer lemma.** The following lemma will allow us to transfer any results established with the $B(f;x)$ normalization to corresponding results with a $\sigma(f;x)$ normalization.

Lemma 9.5. *Let f be a strongly additive function such that $0\leqslant f(p)\leqslant O(1)$ and $B(f;x)\to\infty$. Let Ψ be a distribution function. Suppose that there is an $a>0$ such that $\Psi(a)-\Psi(0)=1$. We have, uniformly in $1\leqslant\Delta\leqslant o(\sigma(f;x))$,*

$$\begin{aligned}
\mathcal{D}_f^{\times}(x;\Delta) &\sim \mathcal{D}_f(x;\Delta) \\
\mathbb{P}\left(\mathcal{Z}_\Psi\left(\sigma^2(f;x)\right)\geqslant\Delta\sigma(f;x)\right) &\sim \mathbb{P}\left(\mathcal{Z}_\Psi\left(B^2(f;x)\right)\geqslant\Delta B(f;x)\right)
\end{aligned}$$

Proof. Both results are consequences of lemma 9.3 and lemma 9.2 respectively. Let us first prove that $\mathcal{D}_f^{\times}(x;\Delta)\sim\mathcal{D}_f(x;\Delta)$ holds. Note that

$$\mathcal{D}_f^{\times}(x;\Delta\cdot\sigma(f;x)/B(f;x)) = \mathcal{D}_f(x;\Delta)$$

Therefore using the asymptotic of Lemma 9.2 we conclude that

$$\mathcal{D}_f(x;\Delta) \sim \exp\left(-\frac{\sigma^3}{B^3}\cdot\frac{\Delta^3}{B}\sum_{k\geqslant 0}\frac{\lambda_f(x;k+2)}{k+3}\cdot\left(\Delta\sigma/B^2\right)^k\right)\int_\Delta^\infty e^{-u^2/2}\cdot\frac{du}{\sqrt{2\pi}} \tag{9.2}$$

where we used the abbreviation $B := B(f;x)$ and $\sigma := \sigma(f;x)$. Further the coefficients $\lambda_f(x;k+2)$ are defined in Lemma 9.2, and satisfy $|\lambda_f(x;k)| \ll C^k$ for some fixed $C > 0$ depending only on f. Because of that bound on $\lambda_f(x;k)$, the function

$$\mathcal{G}_f(x;z) \quad = \quad \sum_{k\geqslant 0}\frac{\lambda_f(x;k+2)}{k+3}\cdot z^k$$

is analytic in $|z| < 1/C$ for all $x > 0$. Therefore

$$\mathcal{G}_f(x;\Delta/B\cdot\sigma/B) \quad = \quad \mathcal{G}_f\left(x;\Delta/B\right) + \left(\Delta/B\cdot(\sigma/B-1)\right)\cdot\mathcal{G}_f'(x;\xi) \tag{9.3}$$

for some $\Delta/B\cdot\sigma/B \leqslant \xi \leqslant \Delta/B$. Derivatives are always taken with respect to the second argument – that is $\mathcal{G}_f'(x;\xi) := (d/d\xi)\mathcal{G}_f(x;\xi)$. Upon using the inequality $\lambda_f(x;k) \ll C^k$ we find the bound $\mathcal{G}_f'(x;\xi) \ll (1 - C\Delta/B)^{-1}$ which is $O(1)$ because $\Delta \leqslant o(B(f;x))$. Further $\Delta/B(1-\sigma/B) \ll \Delta/B^3$ because $\sigma^2 = B^2 + O(1)$ hence $\sigma/B = 1 + O(B^{-2})$. It now follows from (9.3) that $\mathcal{G}_f(x;\Delta/B\cdot\sigma/B) = \mathcal{G}_f(\Delta/B) + O\left(\Delta/B^3\right)$. Note also that $\mathcal{G}_f(x;\Delta/B) \ll 1$ uniformly in $1 \leqslant \Delta \leqslant o(B)$. Using these two estimates, we find that

$$\begin{aligned}
& -(\sigma/B)^3\cdot(\Delta^3/B)\cdot\mathcal{G}_f(x;\Delta/B\cdot\sigma/B) \\
= \quad & -(\sigma/B)^3\cdot(\Delta^3/B)\cdot\left(\mathcal{G}_f(x;\Delta/B) + O\left(\Delta/B^3\right)\right) \\
= \quad & -\left(1+O\left(1/B^2\right)\right)\cdot\Delta^3/B\cdot\mathcal{G}_f(x;\Delta/B) + O\left((\Delta/B)^4\right) \\
= \quad & -\Delta^3/B\cdot\mathcal{G}_f(x;\Delta/B) + O\left((\Delta/B)^3 + (\Delta/B)^4\right)
\end{aligned}$$

Since $\Delta \leqslant o(\sigma(f;x))$ the error term is $o(1)$. The previous equation, together with (9.2) leads to

$$\begin{aligned}
\mathcal{D}_f(x;\Delta) \quad & \sim \quad \exp\left(-(\sigma/B)^3\cdot(\Delta^3/B)\cdot\mathcal{G}_f(x;\Delta/B\cdot\sigma/B)\right)\cdot(1-\Phi(\Delta)) \\
& \sim \quad \exp\left(-(\Delta^3/B)\cdot\mathcal{G}_f(x;\Delta/B)\right)\cdot(1-\Phi(\Delta)) \ \sim\ \mathcal{D}_f^{\times}\left(x;\Delta\right)
\end{aligned}$$

uniformly in $1 \leqslant \Delta \leqslant o(\sigma(f;x))$ and where $1-\Phi(\Delta) := \int_\Delta^\infty e^{-u^2/2}\cdot du$. The above equation establishes the first part of the lemma. The proof of the second part is along similar lines, but easier. Denote by

$$\mathcal{G}_\Psi\left(z\right) := \sum_{k\geqslant 0}\frac{\Lambda(\Psi;k+2)}{k+3}\cdot z^k$$

with the coefficients $\Lambda(\Psi;k)$ defined as in Lemma 9.3. The coefficients $\Lambda(\Psi;k)$ are bounded by C^k, for some suitable $C > 0$, therefore $\mathcal{G}_\Psi(z)$ is analytic in $|z| < 1/C$. Hence $\mathcal{G}_\Psi\left(\Delta/B\right) = \mathcal{G}_\Psi\left(\Delta/\sigma\right) + O\left(\Delta/B - \Delta/\sigma\right) = \mathcal{G}_\Psi\left(\Delta/\sigma\right) + O\left(\Delta/\sigma^3\right)$, where in the error term we used the estimate $B = \sigma + O(1/\sigma)$. Therefore

$$\begin{aligned} -(\Delta^3/B)\cdot\mathcal{G}_\Psi(\Delta/B) &= -(\Delta^3/B)\cdot\left(\mathcal{G}_\Psi(\Delta/\sigma) + O\left(\Delta/\sigma^3\right)\right) \\ &= -(\sigma/B)\cdot(\Delta^3/\sigma)\cdot\mathcal{G}_\Psi(\Delta/\sigma) + O\left((\Delta/\sigma)^4\right) \\ &= -\left(1 + O(1/\sigma^2)\right)\cdot(\Delta^3/\sigma)\cdot\mathcal{G}_\Psi(\Delta/\sigma) + O\left((\Delta/\sigma)^4\right) \\ &= -(\Delta^3/\sigma)\cdot\mathcal{G}_\Psi(\Delta/\sigma) + O\left((\Delta/\sigma)^4 + (\Delta/\sigma)^3\right) \end{aligned}$$

and the error term is $o(1)$ because $\Delta \leqslant o(\sigma(f;x))$. Therefore, using lemma 9.3 we conclude that

$$\begin{aligned} \mathbb{P}\left(\mathcal{Z}_\Psi(\sigma) \geqslant \Delta\sigma\right) &\sim \exp\left(-(\Delta^3/\sigma)\cdot\mathcal{G}_\Psi(\Delta/\sigma)\right)\cdot(1-\Phi(\Delta)) \\ &\sim \exp\left(-(\Delta^3/B)\cdot\mathcal{G}_\Psi(\Delta/B)\right)\cdot(1-\Phi(\Delta)) \sim \mathbb{P}\left(\mathcal{Z}_\Psi(B) \geqslant \Delta B\right) \end{aligned}$$

uniformly in $1 \leqslant \Delta \leqslant o(\sigma(f;x))$ and where $1 - \Phi(\Delta) = \int_\Delta^\infty e^{-u^2/2}\cdot du/\sqrt{2\pi}$. The above equation establishes the second part of the lemma. ☐

9.3. **Proof of the "integers to primes" theorem.**

Proof of Theorem 2.6. By assumptions $\mathcal{D}_f(x;\Delta) \sim \mathbb{P}(\mathcal{Z}_\Psi(\sigma^2(f;x)) \geqslant \Delta\sigma(f;x))$ holds throughout $1 \leqslant \Delta \leqslant o(\sigma(f;x))$. Thus, by lemma 9.5, $\mathcal{D}_f^\times(x;\Delta) \sim \mathbb{P}(\mathcal{Z}_\Psi(B^2(f;x)) \geqslant \Delta B(f;x))$ uniformly in $1 \leqslant \Delta \leqslant o(B(f;x))$. We are going to work with this last condition.

The proof is in three steps. Retaining the notation of Lemma 9.3 and Lemma 9.2 we first show that $\lambda_f(k;x) \longrightarrow \Lambda(\Psi;k)$ for all $k \geqslant 2$ (for $k = 1$ this is trivial). Then, we deduce from there that

$$\frac{1}{B^2(f;x)} \sum_{\substack{p \leqslant x \\ f(p) \leqslant t}} \frac{f(p)^{k+2}}{p} \longrightarrow \int_{\mathbb{R}} t^k d\Psi(t) \tag{9.4}$$

Finally by the method of moments (and an elementary manipulation)

$$\frac{1}{\sigma^2(f;x)} \sum_{\substack{p \leqslant x \\ f(p) \leqslant t}} \frac{f(p)^2}{p}\cdot\left(1 - \frac{1}{p}\right) \longrightarrow \Psi(t) \tag{9.5}$$

The last step being the easy one. To prove our first step we will proceed by induction on $k \geqslant 0$. We will prove the stronger claim that

$$\lambda_f(x;k+2) = \Lambda(\Psi;k+2) + O_k\left(B^{-2^{-(k+1)}}\right)$$

where we write $B = B(f;x)$ to simplify notation. By Lemma 9.2 and 9.3, our assumption $\mathcal{D}_f^\times(x;\Delta) \sim \mathbb{P}(\mathcal{Z}_\Psi(B^2(f;x)) \geqslant \Delta B(f;x))$ (for $1 \leqslant \Delta \leqslant o(B(f;x))$) reduces to

$$-\frac{\Delta^3}{B}\sum_{m\geqslant 0}\frac{\lambda_f(x;m+2)}{m+3}\cdot(\Delta/B)^m = -\frac{\Delta^3}{B}\sum_{m\geqslant 0}\frac{\Lambda(\Psi;m+2)}{m+3}\cdot(\Delta/B)^m + o(1) \tag{9.6}$$

valid throughout the range $1 \leqslant \Delta \leqslant o(\sigma(f;x))$. Let us first establish the base case $\lambda_f(x;2) = \Lambda(\Psi;2) + O(B^{-1/2})$. In (9.6) we choose $\Delta = B^{1/2}$. Because of the bounds $|\lambda_f(x;m)| \leqslant C^m$ and $|\Lambda(\Psi;m)| \leqslant C^m$ (see lemma 9.2 and 9.3) the terms $m \geqslant 1$ contribute $O(1)$. The $m = 0$ term is $\asymp B^{1/2}$. It follows that $\lambda_f(x;2) = \Lambda(\Psi;2) + O(B^{-1/2})$ and so the base case follows. Let us now suppose that for all $\ell < k$, $(k \geqslant 1)$

$$\lambda_f(x;\ell+2) = \Lambda(\Psi;\ell+2) + O_\ell\left(B^{-2^{-(\ell+1)}}\right)$$

Note that we can assume (in the above equation) that the implicit constant depends on k, by taking the max of the implicit constants in $O_\ell(B^{-2^{-(\ell+1)}})$ for $\ell < k$. In equation (9.6) let's choose $\Delta = B^{1-2^{-(k+1)}}$. With this choice of Δ the terms that are $\geqslant k+1$ in (9.6) contribute at most

$$\left(\Delta^3/B\right) \cdot (C \cdot \Delta/B)^{k+1} \ll_k B^2 \cdot B^{-3 \cdot 2^{-(k+1)}} \cdot B^{-(k+1) \cdot 2^{-(k+1)}}$$

on both sides of (9.6). On the other hand, we see (by using the induction hypothesis) that the terms $m \leqslant k-1$ on the left and the right hand side of (9.6) differ by no more that

$$\begin{aligned} & -\frac{\Delta^3}{B} \cdot \left(\sum_{\ell<k} \frac{(\Delta/B)^\ell}{\ell+3} \cdot (\lambda_f(x;\ell+2) - \Lambda(\Psi;\ell+2))\right) \\ = \;& O_k\left(B^2 \cdot B^{-3 \cdot 2^{-(k+1)}} \cdot \left(\sum_{\ell<k} \frac{B^{-\ell \cdot 2^{-(k+1)}}}{\ell+3} \cdot B^{-2^{-(\ell+1)}}\right)\right) \\ = \;& O_k\left(B^2 \cdot B^{-3 \cdot 2^{-(k+1)}} \cdot \sum_{\ell<k} \frac{1}{\ell+3} \cdot B^{-2^{-(k+1)} \cdot \left(\ell + 2^{k-\ell}\right)}\right) \end{aligned}$$

Note that for each integer $\ell < k$ we have $\ell + 2^{k-\ell} \geqslant k+1$. Therefore the above error term is bounded by $O_k(B^2 \cdot B^{-3 \cdot 2^{-k}} \cdot B^{-(k+1)2^{-(k+1)}})$. With these two observations at hand, relation (9.6) reduces to

$$-\frac{\Delta^3}{B} \cdot \frac{(\Delta/B)^k}{k+3}\left[\lambda_f(x;k+2) - \Lambda(\Psi;k+2)\right] = O_k\left(B^{2-3 \cdot 2^{-(k+1)}} \cdot B^{-(k+1) \cdot 2^{-(k+1)}}\right) + o(1)$$

where $\Delta = B^{1-2^{-(k+1)}}$. Dividing by $\Delta^3/B \cdot (\Delta/B)^k \asymp B^{2-3 \cdot 2^{-(k+1)}} \cdot B^{-k \cdot 2^{-(k+1)}}$ on both sides, we conclude that $\lambda_f(x;k+2) - \Lambda(\Psi;k+2) = O_k(B^{-2^{-(k+1)}})$ as desired, thus finishing the inductive step. Now, we will prove that $\lambda_f(x;k) \longrightarrow \Lambda(\Psi;k)$ implies

$$\mathcal{M}_f(x;\ell) := \frac{1}{B^2(f;x)} \sum_{p \leqslant x} \frac{f(p)^{\ell+2}}{p} \longrightarrow \int_{\mathbb{R}} t^\ell d\Psi(t) \tag{9.7}$$

for each fixed $\ell \geqslant 0$. This follows almost immediately from the recurrence relation for $\lambda_f(x;k)$ and $\Lambda(\Psi;k)$. Indeed let us prove (9.7) by induction on $k \geqslant 0$. The base case $k = 0$ is obvious, for the left hand side and right hand side of (9.7) are both equal to 1. Let us now suppose that (9.7) holds for all $\ell < k$. We will prove that convergence also holds for $\ell = k$. By definition of $\lambda_f(x;k+1)$ we have

$$\lambda_f(x;k+1) = -\sum_{j=2}^{k} \frac{\mathcal{M}_f(x;j-1)}{j!} \sum_{\ell_1+\ldots+\ell_j=k+1} \lambda_f(x;\ell_1)\ldots\lambda_f(x;\ell_j) - \frac{\mathcal{M}_f(x;k)}{(k+1)!} \tag{9.8}$$

(we single out $j = k+1$ on the right hand side). By induction hypothesis $\mathcal{M}_f(x;j-1) \longrightarrow \int t^{j-1}d\Psi(t)$ as $x \to \infty$, for $j \leqslant k$. Further as we've shown earlier $\lambda_f(x;i) \longrightarrow \Lambda(\Psi;i)$ for all $i \geqslant 0$. Therefore the whole double sum on the right hand side of (9.8) tends to

$$-\sum_{j=2}^{k} \frac{1}{j!} \int_{\mathbb{R}} t^{j-1}d\Psi(t) \sum_{\ell_1+\ldots+\ell_j=k+1} \Lambda(\Psi;\ell_1)\cdot\ldots\cdot\Lambda(\Psi;\ell_j)$$

which, by definition of $\Lambda(\Psi;k)$ is equal to $\Lambda(\Psi;k+1) + 1/(k+1)! \int_{\mathbb{R}} t^k d\Psi(t)$. But also $\lambda_f(x;k+1) \longrightarrow \Lambda(\Psi;k+1)$ because $\lambda_f(x;i) \longrightarrow \Lambda(\Psi;i)$ for all $i \geqslant 0$. Thus the left hand side of (9.8) tends to $\Lambda(\Psi;k+1)$ while the double sum on the right hand side of (9.8) tends to $\Lambda(\Psi;k+1) + 1/(k+1)! \int_{\mathbb{R}} t^k d\Psi(t)$. Therefore equation (9.8) transforms into

$$\Lambda(\Psi;k+1) = \Lambda(\Psi;k+1) + \frac{1}{(k+1)!}\int_{\mathbb{R}} t^k d\Psi(t) - \frac{\mathcal{M}_f(x;k)}{(k+1)!} + o_{x\to\infty}(1)$$

and $\mathcal{M}_f(x;k) \to \int_{\mathbb{R}} t^k d\Psi(t)$ $(x \to \infty)$ follows. This establishes the induction step and thus (9.7) for all fixed $\ell \geqslant 0$. Now we use the method of moments to prove that

$$F(x;t) := \frac{1}{B^2(f;x)} \sum_{\substack{p \leqslant x \\ f(p) \leqslant t}} \frac{f(p)^2}{p} \underset{x\to\infty}{\longrightarrow} \Psi(t)$$

holds at all continuity points t of $\Psi(t)$. Let us note that, the k-th moment of the distribution function $F(x;t)$, is given by $\mathcal{M}_f(x;k)$, and as we've just shown this converges to the the k-th moment of $\Psi(t)$. That is

$$\int_{\mathbb{R}} t^k dF(x;t) = \frac{1}{B^2(f;x)} \sum_{p\leqslant x} \frac{f(p)^{k+2}}{p} \longrightarrow \int_{\mathbb{R}} t^k d\Psi(t)$$

Since $\Psi(a) - \Psi(0) = 1$ for some $a > 0$, the distribution function Ψ satisfies the assumption of Lemma 9.4, hence, by Lemma 9.4 (the method of moments) we have $F(x;t) \longrightarrow \Psi(t)$ at all continuity points t of Ψ. Finally, we deduce that

$$\frac{1}{\sigma^2(f;x)} \sum_{\substack{p \leqslant x \\ f(p) \leqslant t}} \frac{f(p)^2}{p} \cdot \left(1 - \frac{1}{p}\right) \longrightarrow \Psi(t) \tag{9.9}$$

at all continuity points of Ψ. This is almost trivial, because $B^2(f;x)$ and $\sigma^2(f;x)$ differ only by an $O(1)$, and the sum $\sum f(p)^2/p^2 = O(1)$ because the $f(p)$ are $O(1)$. Hence, by a simple computation

$$\frac{1}{\sigma^2(f;x)} \sum_{\substack{p \leqslant x \\ f(p) \leqslant t}} \frac{f(p)^2}{p} \cdot \left(1 - \frac{1}{p}\right) \ = \ F(x;t) \ + \ O\left(\frac{1}{B^2(f;x)}\right)$$

And since $F(x;t) \longrightarrow \Psi(t)$ at all continuity points of Ψ, it follows that (9.9) must be true. □

10. Primes to integers

We keep the same notation as in the previous section. Namely we let

$$B^2(f;x) := \sum_{p\leqslant x} \frac{f(p)^2}{p} \quad \text{and} \quad \mathcal{D}_f^{\times}(x;\Delta) := \frac{1}{x} \cdot \# \left\{ n \leqslant x : \frac{f(n) - \mu(f;x)}{B(f;x)} \geqslant \Delta \right\}$$

We first need to modify a little some of the known large deviations results for $\mathcal{D}_f^{\times}(x;\Delta)$ and $\mathbb{P}(\mathcal{Z}_\Psi(B^2(f;x)) \geqslant \Delta B(f;x))$.

10.1. **Large deviations for $\mathcal{D}_f^{\times}(x;\Delta)$ and $\mathbb{P}(\mathcal{Z}_\Psi(x) \geqslant t)$ revisited.** First we require the result of Maciulis ([14], theorem) in a "saddle-point" version.

Lemma 10.1. *Let g be a strongly additive function such that $0 \leqslant g(p) \leqslant O(1)$ and $B(g;x) \to \infty$. We have uniformly in $1 \leqslant \Delta \leqslant o(B(g;x))$,*

$$\mathcal{D}_g^{\times}(x;\Delta) \sim \exp\left(\sum_{p\leqslant x} \frac{e^{\eta g(p)} - \eta g(p) - 1}{p} - \eta \sum_{p\leqslant x} \frac{g(p)(e^{\eta g(p)} - 1)}{p} \right) \frac{e^{\Delta^2/2}}{\sqrt{2\pi}} \int_{\Delta}^{\infty} e^{-t^2/2} dt$$

where $\eta = \eta_g(x;\Delta)$ is defined as the unique positive solution of the equation

$$\sum_{p\leqslant x} \frac{g(p)e^{\eta g(p)}}{p} = \mu(g;x) + \Delta B(g;x)$$

Furthermore $\eta_g(x;\Delta) = \Delta/B(g;x) + O(\Delta^2/B(g;x)^2)$.

Proof. Only the last assertion needs to be proved, because it is not stated explicitly in Maciulius's paper. Fortunately enough, it's a triviality. Indeed, writing $\eta = \eta_g(x;\Delta)$, we find that

$$0 \leqslant \eta_g(x;\Delta) \sum_{p\leqslant x} \frac{g(p)^2}{p} \leqslant \sum_{p\leqslant x} \frac{g(p)(e^{\eta g(p)} - 1)}{p} = \Delta B(g;x)$$

Dividing by $B^2(g;x)$ on both sides $0 \leqslant \eta_g(x;\Delta) \leqslant \Delta/B(g;x)$ follows. Now expanding $e^{\eta g(p)} - 1 = \eta g(p) + O(\eta^2 g(p)^2)$ and noting that $O(\eta^2 g(p)^2) = O(\eta^2 g(p))$ because $g(p) = O(1)$, we find that

$$\Delta B(g;x) = \sum_{p\leqslant x} \frac{g(p)(e^{\eta g(p)} - 1)}{p} = \eta \sum_{p\leqslant x} \frac{g(p)^2}{p} + O\left(\eta^2 \sum_{p\leqslant x} \frac{g(p)^2}{p}\right)$$

Again dividing by $B^2(g;x)$ on both sides, and using the bound $\eta = O(\Delta/B(g;x))$ the claim follows. $\square$

Adapting Hwang's [11] result we prove the following.

Lemma 10.2. *Let Ψ be a distribution function. Suppose that there is an $\alpha > 0$ such that $\Psi(\alpha) - \Psi(0) = 1$. Let $B^2 = B^2(x) \to \infty$ be some function tending to infinity. Then, uniformly in $1 \leqslant \Delta \leqslant o(B(x))$ the quantity $\mathbb{P}\left(\mathcal{Z}_\Psi(B^2) \geqslant \Delta B\right)$ is asymptotic to*

$$\exp\left(B^2 \int_{\mathbb{R}} \frac{e^{\rho u} - \rho u - 1}{u^2} d\Psi(u) - B^2 \cdot \rho \int_{\mathbb{R}} \frac{e^{\rho u} - 1}{u} d\Psi(u)\right) \cdot \frac{e^{\Delta^2/2}}{\sqrt{2\pi}} \int_{\Delta}^{\infty} e^{-t^2/2}\, dt$$

where $\rho = \rho_\Psi(B(x);\Delta)$ is defined implicitly, as the unique positive solution to

$$B^2(x) \int_{\mathbb{R}} \frac{e^{\rho u} - 1}{u} d\Psi(u) = \Delta \cdot B(x)$$

Proof. We keep the same notation as in lemma 9.3. While proving lemma 9.3 we established the following useful relationship (see (9.1))

$$\sum_{k \geqslant 0} \Lambda(\Psi; k+2)\xi^{k+2} = (u')^{-1}(\xi) - \xi \text{ where } u(z) = \int_{\mathbb{R}} \frac{e^{zt} - zt - 1}{t^2} d\Psi(t)$$

Here $(u')^{-1}$ denotes the inverse function of u'. Integrating the above gives

$$\sum_{k \geqslant 0} \frac{\Lambda(\Psi; k+2)}{k+3} \cdot \xi^{k+3} = -\frac{\xi^2}{2} + \xi \cdot (u')^{-1}(\xi) - u((u')^{-1}(\xi))$$

Now choose $\xi = \Delta/B$, then by definition $\rho = \rho_\Psi(B(x);\Delta) = (u')^{-1}(\xi)$. Thus the above formula becomes

$$\sum_{k \geqslant 0} \frac{\Lambda(\Psi; k+2)}{k+3} \cdot (\Delta/B)^{k+3} = -\frac{(\Delta/B)^2}{2} + \frac{\Delta}{B} \cdot \rho - \int_{\mathbb{R}} \frac{e^{\rho t} - \rho t - 1}{t^2} d\Psi(t)$$

Also, note that by definition $\Delta/B = \int_{\mathbb{R}} (e^{\rho t} - 1)/t \cdot d\Psi(t)$. Using the above formula (in which we replace $(\Delta/B) \cdot \rho$ by $\rho \int_{\mathbb{R}} (e^{\rho t} - 1)/t \cdot d\Psi(t)$) and lemma 9.3

$$\mathbb{P}(\mathcal{Z}_\Psi(B^2) \geqslant \Delta B) \sim \exp\left(-B^2 \sum_{k \geqslant 0} \frac{\Lambda(\Psi; k+2)}{k+3} \cdot (\Delta/B)^{k+3}\right) \int_{\Delta}^{\infty} e^{-u^2/2} \cdot \frac{du}{\sqrt{2\pi}}$$

$$= \exp\left(B^2 \int_{\mathbb{R}} \frac{e^{\rho t} - \rho t - 1}{t^2} d\Psi(t) - B^2 \cdot \rho \int_{\mathbb{R}} \frac{e^{\rho t} - 1}{t} d\Psi(t)\right) \cdot \frac{e^{\Delta^2/2}}{\sqrt{2\pi}} \int_{\Delta}^{\infty} e^{-u^2/2} du$$

This is the claim. $\square$

Before we prove Theorem 2.5, we need to show that the parameters $\eta_g(x;\Delta)$ and $\rho_\Psi(B(f;x);\Delta)$ (as defined respectively in Lemma 10.1 and Lemma 10.2) are "close" when the distribution of the $g(p)$'s resembles $\Psi(t)$. The "closeness" assertion is made precise in the next lemma.

Lemma 10.3. *Let $\Psi(\cdot)$ be a distribution function. Let f be a positive strongly additive function. Suppose that $0 \leqslant f(p) \leqslant O(1)$ for all primes p. Let*

$$K_f(x;t) := \frac{1}{B^2(f;x)} \sum_{\substack{p \leqslant x \\ f(p) \leqslant t}} \frac{f(p)^2}{p}$$

If $K_f(x;t) - \Psi(t) \ll 1/B^2(f;x)$ uniformly in $t \in \mathbb{R}$, then

$$\rho_\Psi(B(f;x);\Delta) - \eta_f(x;\Delta) = o(1/B^2(f;x))$$

uniformly in $1 \leqslant \Delta \leqslant o(B(f;x))$. The symbols $\rho_\Psi(B(x);\Delta)$ and $\eta_f(x;\Delta)$ are defined in lemma 10.2 and lemma 10.1 respectively.

Proof. Let $\eta = \eta_f(x;\Delta)$. Recall that by lemma 10.1, $\eta = o(1)$ in the range $1 \leqslant \Delta \leqslant o(B(f;x))$. This will justify the numerous Taylor expansions involving the parameter η. With $K_f(x;t)$ defined as in the statement of the lemma, we have

$$\begin{aligned} \sum_{p\leqslant x} \frac{f(p)e^{\eta f(p)}}{p} - \mu(f;x) &= \sum_{p \leqslant x} \frac{f(p)(e^{\eta f(p)} - 1)}{p} \\ &= B^2(f;x)\int_{\mathbb{R}} \frac{e^{\eta t} - 1}{t} dK_f(x;t) \end{aligned} \tag{10.1}$$

Let $M > 0$ be a real number such that $0 \leqslant f(p) \leqslant M$ for all p. Since the $f(p)$ are bounded, for each $x > 0$ the distribution function $K_f(x;t)$ is supported on $[0;M]$. Furthermore since $K_f(x;t) \to \Psi(t)$ the distribution function $\Psi(t)$ is supported on exactly the same interval. From these considerations, it follows that

$$\begin{aligned} \int_{\mathbb{R}} \frac{e^{\eta t} - 1}{t} dK_f(x;t) &= \int_0^M \frac{e^{\eta t} - 1}{t} dK_f(x;t) \\ = \int_0^M \frac{e^{\eta t} - 1}{t} d\Psi(t) &+ \int_0^M \big(K_f(x;t) - \Psi(t)\big) \cdot \left[\frac{e^{\eta t} - \eta t \cdot e^{\eta t} - 1}{t^2}\right] dt \end{aligned} \tag{10.2}$$

By a simple Taylor expansion $e^{\eta t} - \eta t \cdot e^{\eta t} - 1 = O(\eta^2 t^2)$. Therefore the integral on the right hand side is bounded by $O(\eta^2/B^2(f;x))$. We conclude from (10.1) and (10.2) that

$$\sum_{p \leqslant x} \frac{f(p)e^{\eta f(p)}}{p} - \mu(f;x) = B^2(f;x)\int_{\mathbb{R}} \frac{e^{\eta t} - 1}{t} d\Psi(t) + O\left(\eta^2\right) \tag{10.3}$$

By definition of ρ_Ψ and η_f,

$$B^2(f;x)\int_{\mathbb{R}} \frac{e^{\rho_\Psi(B;\Delta)t} - 1}{t} d\Psi(t) = \Delta B(f;x) = \sum_{p \leqslant x} \frac{f(p)e^{\eta_f(x;\Delta)f(p)}}{p} - \mu(f;x) \tag{10.4}$$

From (10.3) and (10.4) it follows that

$$B^2(f;x)\int_{\mathbb{R}} \frac{e^{\rho_\Psi(B;\Delta)t} - e^{\eta_f(x;\Delta)t}}{t} d\Psi(t) = O\left(\eta_f^2(x;\Delta)\right) \tag{10.5}$$

Since $\Psi(t)$ is supported on $[0;M]$ we can restrict the above integral to $[0;M]$. By lemma 10.1, we have $\eta_f(x;\Delta) \sim \Delta/B(f;x) = o(1)$ in the range $\Delta \leqslant o(B(f;x))$. Also $0 \leqslant \rho_\Psi(x;\Delta) \leqslant \int_{[0;M]}(e^{\rho_\Psi(x;\Delta)t} - 1)/t \cdot d\Psi(t) = \Delta/B(f;x) = o(1)$ for Δ in the same range. Write $\rho := \rho_\Psi(B;\Delta)$ and $\eta := \eta_f(x;\Delta)$. For $0 \leqslant t \leqslant M$, we have

$$\begin{aligned}(1/t)\left(e^{\rho t} - e^{\eta t}\right) &= (1/t)e^{\rho t}\cdot(1 - e^{(\eta-\rho)t}) \\ &= e^{\rho t}\cdot(\eta-\rho) + O((\eta-\rho)^2) \asymp \eta - \rho\end{aligned}$$

because $\rho = o(1), \eta = o(1)$. Inserting this estimate into (10.5) we get $\eta - \rho = O(\eta^2/B^2(f;x)) = o(1/B^2(f;x))$ since $\eta^2 = o(1)$. The lemma is proved. □

10.2. **Proof of the "primes to integers" theorem.**

Proof of Theorem 2.5. Note that

$$\frac{1}{\sigma^2(f;x)} \sum_{\substack{p \leqslant x \\ f(p) \leqslant t}} \frac{f(p)^2}{p}\cdot\left(1 - \frac{1}{p}\right) = \frac{1}{B^2(f;x)} \sum_{\substack{p \leqslant x \\ f(p) \leqslant t}} \frac{f(p)^2}{p} + O\left(\frac{1}{B^2(f;x)}\right)$$

and denote the main term on the right hand side by $K_f(x;t)$. By assumption, the left hand side in the above equation, differs from $\Psi(t)$ by$\ll 1/\sigma^2(f;x) \asymp 1/B^2(f;x)$. Hence $K_f(x;t) = \Psi(t) + O(1/B^2(f;x))$. Let $\eta = \eta_f(x;\Delta)$ be the parameter from lemma 10.1. Proceeding as in the proof of the previous lemma, we get

$$\sum_{p\leqslant x} \frac{e^{\eta f(p)} - \eta f(p) - 1}{p} = B^2(f;x)\int_{\mathbb{R}} \frac{e^{\eta u} - \eta u - 1}{u^2} d\Psi(u) + o(1) \tag{10.6}$$

$$\eta\sum_{p\leqslant x} \frac{f(p)\cdot(e^{\eta f(p)} - 1)}{p} = B^2(f;x)\eta\int_{\mathbb{R}} \frac{e^{\eta u} - 1}{u} d\Psi(u) + o(1) \tag{10.7}$$

throughout the range $1 \leqslant \Delta \leqslant o(B(f;x))$. Let $\rho := \rho_\Psi(B(f;x);\Delta)$ denote the parameter from Lemma 10.2. The functions on the right of (10.6) and (10.7) are analytic. Therefore, by lemma 10.3,

$$B^2(f;x)\int_{\mathbb{R}} \frac{e^{\eta u} - \eta u - 1}{u^2} d\Psi(u) = B^2(f;x)\int_{\mathbb{R}} \frac{e^{\rho u} - \rho u - 1}{u^2} d\Psi(u) + o(1) \tag{10.8}$$

$$B^2(f;x)\eta\int_{\mathbb{R}} \frac{e^{\eta u} - 1}{u} d\Psi(u) = B^2(f;x)\rho\int_{\mathbb{R}} \frac{e^{\rho u} - 1}{u} d\Psi(u) + o(1) \tag{10.9}$$

uniformly in $1 \leqslant \Delta \leqslant o(B(f;x))$. On combining (10.6) with (10.8) and (10.7) with (10.9) we obtain

$$\sum_{p \leqslant x} \frac{e^{\eta f(p)} - \eta f(p) - 1}{p} - \eta \sum_{p \leqslant x} \frac{f(p)(e^{\eta f(p)} - 1)}{p}$$
$$= B^2(f;x) \int_{\mathbb{R}} \frac{e^{\rho u} - \rho u - 1}{u^2} d\Psi(u) - B^2(f;x)\rho \int_{\mathbb{R}} \frac{e^{\rho u} - 1}{u} d\Psi(u) + o(1)$$

By lemma 10.1, lemma 10.2 and the above equation, we get

$$\mathcal{D}_f^{\times}(x;\Delta) \sim \mathbb{P}\left(\mathcal{Z}_{\Psi}\left(B^2(f;x)\right) \geqslant \Delta B(f;x)\right)$$

uniformly in $1 \leqslant \Delta \leqslant o(B(f;x))$. By lemma 9.5, it follows that

$$\mathcal{D}_f(x;\Delta) \sim \mathbb{P}\left(\mathcal{Z}_{\Psi}(\sigma^2(f;x)) \geqslant \Delta\sigma(f;x)\right)$$

uniformly in $1 \leqslant \Delta \leqslant o(\sigma(f;x))$. This proves the theorem. □

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